Substitutions over $\mathbb{C}$-powered symbols, and Rauzy fractals for imaginary directions

by

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Abstract

We introduce substitutions over complex powered symbols $\mathtt{a}^{\mathtt{z}}(\mathtt{z}\in\mathbb{C})$, and extend the definition of the Rauzy fractal and give some new examples of Rauzy fractals. We also give some problems and results related to simultaneous Diophantine approximations and multidimensional complex continued fractions.

§1. Introduction

Let $\mathscr{A}$ be an alphabet. In formal language theory, for a symbol $\mathtt{a}\in\mathscr{A}$ and an integer $n\in\mathbb{Z}_{\geq 0}$, $\mathtt{a}^{n}$ ($\mathtt{a}^{0}:=\lambda$) denotes a word of length n consisting of n identical symbols $\mathtt{a}$, where λ is the empty word. In Section 1, we introduce a complex powered symbol $\mathtt{a}^{\mathtt{z}}$ for a complex number $\mathtt{z}\in\mathbb{C}$, which is considered as a symbol having $\mathtt{z}$ as its quantity (fractional powered symbol $\mathtt{a}^{t}$ ($t>0$, $t\notin\mathbb{Z}$) has been introduced in [T2], [Ka]). In particular, usual symbols are considered as symbols having 1 as their quantity. In Section 3, we introduce $\mathbb{C}$-substitutions, which are substitutions over complex powered symbols, and give some lemmas related to fixed points of $\mathbb{C}$-substitutions. We give an equvalence class of $\mathbb{C}$-substitutions and related lemmas in Section 4, which play important roles for the definition of the Rauzy set. In Section 5, we give a lemma related to the powers of a matrix with complex entries, and three theorems concerning simultaneous approximations of a vector$\in\mathbf{L}^{s}$ by vectors$\in\mathbf{K}^{s}$, where s+1 is the number of elements of $\mathscr{A}$, $\mathbf{K}=\mathbf{K}(\sigma)$ is a number field possibly imaginary and transcendental determined by a $\mathbb{C}$-substitution σ, $\mathbf{E}=\mathbf{E}(\sigma)$ is a subfield of $\mathbf{K}$ generated by the entries of the incidence matrix of σ, and $\mathbf{L}=\mathbf{L}(\sigma)$ is an algebraic extension of the subfield $\mathbf{E}$ of $\mathbf{K}$ of degree $[\mathbf{L}:\mathbf{E}]=s+1$, cf. Theorems 7, 9, 11. At the end of Section 5, we give some definitions of Rauzy sets (the so called Rauzy fractals) for fixed points of $\mathbb{C}$-substitutions, and try to extend the meaning of Rauzy sets, cf. Example 9 in Section 7. In Section 6, we show a connection between $\mathbb{C}$-substitutions and multidimensional continued fractions with denominators$\in\mathbf{K}^{s}$ by introducing a transposed value of a continued fraction, cf. Theorem 15. In Section 8, we give some other definitions of $\mathbb{C}$-substitutions for which the multiplicity $M_{\sigma\tau}=M_{\sigma}M_{\tau}$ holds, cf. Lemma 1, Section 3. The main objective of this paper is to show, through some examples of Rauzy sets for imaginary directions, that $\mathbb{C}$-substitutions still work fantastically as well as usual substitutions over $\mathscr{A}$ (cf. Section 7), and to give some problems concerning $\mathbb{C}$-substitutions (cf. Sections 5-8). By introducing dual substitutions of $\mathbb{C}$-substitutions, we can consider stepped surfaces for imaginary directions, cf. [TY2].

§2. Notation

Let $\mathscr{A}$ be an alphabet. We mean by an alphabet a nonempty finite set of symbols unless otherwise mentioned. Throughout the paper, s denotes a positive integer, s+1 denotes the number of elements of $\mathscr{A}$, $\underline{\underline{\gamma}}$ denotes an (s+1)-dimensional vector$\in\mathbb{C}^{s+1}$ and $\underline{\gamma}$ denotes an s-dimensional vector$\in\mathbb{C}^{s}$. We write $\mathscr{A}=\{\mathtt{a}_0, \mathtt{a}_1, ..., \mathtt{a}_s\}$, and in some contexts, we write $\mathscr{A}=\{\mathtt{a}, \mathtt{b}, ...\}$. We consider complex powered symbols $\mathtt{a}^{\mathtt{z}}$ for $\mathtt{a}\in\mathscr{A}$, $\mathtt{z}\in\mathbb{C}$. At present, we give no meaning for them, we just consider complex powered symbols in formal sense and suppose the following:

Axiom (i) $a^0=\lambda$, $a^1=a$ for all $a\in\mathscr{A}$, (ii) $a^z=b^w$ ($a, b\in\mathscr{A}$, $z, w\in\mathbb{C}$) $\Longrightarrow$ ($a=b$ & $z=w$) or $z=w=0$ ($a, b\in\mathscr{A}$, z, $w\in\mathbb{C}$).

The axiom implies

$$a^z\neq\lambda \text{ for all } a\in\mathscr{A} \text{ and } z\in\mathbb{C}^\times:=\mathbb{C}\setminus\{0\}.$$

We put

$$\mathscr{A}^\wedge\mathbb{C}^\times:=\{a^z;\ a\in\mathscr{A},\ z\in\mathbb{C}^\times\},$$

in particular, we write

$$a^\wedge\mathbb{C}^\times:=\{a^z\ ;\ z\in\mathbb{C}^\times\}.$$

A nonempty finite word over $\mathscr{A}^\wedge\mathbb{C}^\times$ is a map

$$w:\ \{1, 2, ..., n\}\longrightarrow\mathscr{A}^\wedge\mathbb{C}^\times\ (n\in\mathbb{Z}_{>0}).$$

We denote by $(\mathscr{A}^\wedge\mathbb{C}^\times)^{\{1,2,...,n\}}$ the set of all maps w: $\{1,2,...,n\}\longrightarrow\mathscr{A}^\wedge\mathbb{C}^\times$. For a word $w\in(\mathscr{A}^\wedge\mathbb{C}^\times)^{\{1,2,...,n\}}$, we

say that w is a word of symbolic length n, which is denoted by $|w|_{symb}$. We write

$$w=w(1)w(2)\cdots w(n)=w_1^{z_1}w_2^{z_2}\cdots w_n^{z_n}$$

for $w\in(\mathscr{A}^\wedge\mathbb{C}^\times)^{\{1,2,...,n\}}$ with $w(j)=w_j^{z_j}$ ($w_j\in\mathscr{A}$, $z_j\in\mathbb{C}^\times$, $j\in\{1,2, ..., n\}$). For a finite word $w=w_1^{z_1}w_2^{z_2}\cdots w_n^{z_n}\in(\mathscr{A}^\wedge\mathbb{C}^\times)^{\{1,2,...,n\}}$ ($w_j\in\mathscr{A}$, $z_j\in\mathbb{C}^\times$), we put

$$|w|_a:=\text{the sum of the complex numbers } z_k \text{ such that } w_k=a\ (1\le k\le n)$$

for each $a\in\mathscr{A}$.

In what follows, we denote by $\sqrt{-1}$ the imaginary unit. For example, $u=a^5a^{-1}b^{2/3+3\sqrt{-1}}b^{5/3-2\sqrt{-1}}a$ is a word over $\{a, b\}^\wedge\mathbb{C}^\times$ of symbolic length 5 determined by $u\in(\mathscr{A}^\wedge\mathbb{C}^\times)^{\{1,2,...,5\}}$ and $u(1)=a^5$, $u(2)=a^{-1}$, $u(3)=b^{2/3+3\sqrt{-1}}$,

$u(4)=b^{5/3-2\sqrt{-1}}$, $u(5)=a$. On the other hand, $v=a^4b^{7/3+\sqrt{-1}}a$ is a word of symbolic length 3, so that $u=a^5a^{-1}b^{2/3+3\sqrt{-1}}b^{5/3-2\sqrt{-1}}a\neq v=a^4b^{7/3+\sqrt{-1}}$, and $(|u|_a, |u|_b)=(|v|_a, |v|_b)=(5, 7/3+\sqrt{-1})$. The metrical length of $w=w_1^{z_1}w_2^{z_2}\cdots w_n^{z_n}\in(\mathscr{A}^\wedge\mathbb{C}^\times)^{\{1,2,...,n\}}$ ($w_j\in\mathscr{A}$, $z_j\in\mathbb{C}^\times$) written as $|w|_{metr}$ is defined by

$$|w|_{metr}=|z_1|+|z_2|+\cdots+|z_n|,$$

where $|z|$ ($z\in\mathbb{C}^\times$) is the usual absolute value of a complex number z. We mean by $\mathbb{C}^{s+1}$ the unitary space, i.e., the metric vector space of dimension s+1 over $\mathbb{C}$ with the metric

$$d(\underline{\underline{u}}, \underline{\underline{v}}):=\|\underline{\underline{u}}-\underline{\underline{v}}\|,\ \|\underline{\underline{u}}\|:=\sqrt{(\underline{\underline{u}}, \underline{\underline{u}})},$$

here, $(\underline{\underline{u}}, \underline{\underline{v}})$ is the Hermitian product defined by

$$(\underline{\underline{u}}, \underline{\underline{v}}):=\Sigma_{0\le j\le s}u_j\overline{v_j} \text{ for } \underline{\underline{u}}=(u_0, ..., u_s),\ \underline{\underline{v}}=(v_0, ..., v_s)\in\mathbb{C}^{s+1},$$

$$\overline{z}=\text{the complex conjugate of } z\in\mathbb{C}.$$

We denote by $\underline{\underline{e_0}}=(1,0,0, ...,0)$, $\underline{\underline{e_1}}=(0,1,0 ...,0)$, ..., $\underline{\underline{e_s}}=(0,0, ...,0,1)\in\mathbb{C}^{s+1}$ the canonical basis of $\mathbb{C}^{s+1}$. For a finite word

$$w=w_1^{z_1}w_2^{z_2}\cdots w_n^{z_n}\in(\mathscr{A}^\wedge\mathbb{C}^\times)^{\{1,2,...,n\}}\ (w_j\in\mathscr{A}=\{a_0, a_1, ..., a_s\},\ z_j\in\mathbb{C}^\times)$$

we define $PR(w)=\{\underline{\underline{p}}(w, 1), \underline{\underline{p}}(w, 2), ..., \underline{\underline{p}}(w, n)\}\subset\mathbb{C}^{s+1}$ by

$$\underline{\underline{p}}(w, 0):=(0,0,...,0)\in\mathbb{C}^{s+1},\ \underline{\underline{p}}(w, m):=\underline{\underline{p}}(w, m-1)+z_m\cdot\underline{\underline{e_j}} \text{ if } w_m=a_j\ (1\le m\le n),$$

and LR(w) by

$$LR(w):=\bigcup_{1\le m\le n}(\underline{\underline{p}}(w, m-1), \underline{\underline{p}}(w, m)]$$

with

$$(\underline{\underline{p}}(w, m-1), \underline{\underline{p}}(w, m)]:=\{\underline{\underline{p}}(w, m-1)+tz_m\cdot\underline{\underline{e_j}};\ 0<t\le 1\} \text{ if } w_m=a_j\ (1\le m\le n).$$

The set PR(w) is a finite set of points, and LR(w) is a broken line consisting of the segments $(\underline{\underline{p}}(w, m-1), \underline{\underline{p}}(w, m)]$ $(1\leq m\leq n)$ in the unitary space $\in\mathbb{C}^{s+1}$, which are considered as realizations/representations of a finite word w over $\mathcal{A}^\wedge\mathbb{C}^\times$, and will be called as the point-set representation (resp., the line representation) of w. The set of all finite words (resp., an infinite word) over $\mathcal{A}^\wedge\mathbb{C}^\times$ is denoted by $(\mathcal{A}^\wedge\mathbb{C}^\times)^*$ (resp., $(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$). The set $(\mathcal{A}^\wedge\mathbb{C}^\times)^*$ becomes a free monoid generated by the set $\mathcal{A}^\wedge\mathbb{C}^\times$ with respect to the concatenation with λ as its unit. It is clear that

$$|uv|_{symb} = |u|_{symb} + |v|_{symb},\ |uv|_a = |u|_a + |v|_a (\forall a\in\mathcal{A}),\ |uv|_{metr} = |u|_{metr}+ |v|_{metr}$$

where $uv\in(\mathcal{A}^\wedge\mathbb{C}^\times)^*$ is a word obtained by the concatenation of $u\in(\mathcal{A}^\wedge\mathbb{C}^\times)^*$ and $v\in(\mathcal{A}^\wedge\mathbb{C}^\times)^*$, i.e., uv is a map

$$uv:\ \{1, 2, ..., k+h\} \longrightarrow \mathcal{A}^\wedge\mathbb{C}^\times$$

defined by $(uv)(n):=u(n)$ $(1\leq n\leq k)$, $(uv)(n):=v(n-k)$ $(k+1\leq n\leq k+h)$ for $|u|_{symb}=k$, $|v|_{symb}=h$. We can also define the concatenation $uv=w(1)w(2)w(3)\cdots$ of u and v for $u\in(\mathcal{A}^\wedge\mathbb{C}^\times)^{\{1,2,\ldots,n\}}$ and $v\in(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$ by

$$w(m):=u(m)\ (1\leq m\leq n),\ w(m):=v(m-n)\ (m\geq n+1).$$

We say that $u\in(\mathcal{A}^\wedge\mathbb{C}^\times)^*$ is a prefix of $w\in(\mathcal{A}^\wedge\mathbb{C}^\times)^*\bigcup(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$ if w=uv for a word $v\in(\mathcal{A}^\wedge\mathbb{C}^\times)^*\bigcup(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$. In other words, we consider words over an "alphabet" $\mathcal{A}^\wedge\mathbb{C}^\times$ having continuum cardinality, and consider subwords (prefixes, etc.) in usual sense. Thus we can define the subwords$\in(\mathcal{A}^\wedge\mathbb{C}^\times)^*$ of an infinite words as well as for usual words over $\mathcal{A}$. The symbolic length of an infinite word w is, by definition, $|w|_{symb} = \infty$. To be precise, we should say that $w\in(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$ is a symbolically infinite words instead of saying that w is an infinite word.

For an infinite word $w=w_1^{z_1} w_2^{z_2}\cdots w_n^{z_n}\cdots\in(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$ $(w_j\in\mathcal{A}, z_j\in\mathbb{C}^\times)$, we can extend the definitions $|w|_{metr}$, PR(w), LR(w) given above:

$$|w|_{metr} := |z_1|+|z_2|+\cdots+|z_n|+\cdots\in\mathbb{C}^\times\bigcup\{\infty\};$$

$$\underline{\underline{p}}(w, 0):=(0,0,...,0)\in\mathbb{C}^{s+1},\ \underline{\underline{p}}(w, n):=\underline{\underline{p}}(w, n-1)+z_n\cdot \underline{\underline{e_j}} \text{ if } w_n=a_j\ (n=1, 2, ...),$$

$$PR(w)=\{\underline{\underline{p}}(w, 1), \underline{\underline{p}}(w, 2), ..., \underline{\underline{p}}(w, n), ...\}\subset\mathbb{C}^{s+1}$$

$$(\underline{\underline{p}}(w, n-1), \underline{\underline{p}}(w, n)] :=\{\underline{\underline{p}}(w, n-1)+tz_n\cdot\underline{\underline{e_j}};\ 0<t\leq 1\} \text{ if } w_n=a_j\ (n=1, 2, ...),$$

$$LR(w) :=\bigcup_{n=1,2,3,\ldots} (\underline{\underline{p}}(w, n-1), \underline{\underline{p}}(w, n)].$$

We write

$$w[1,n]:=w(1)w(2)\cdots w(n)\ (w[1,0]:=\lambda)$$

for $w=w(1)w(2)\cdots\in(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$. It is clear that

$$\underline{\underline{p}}(w, n)=(|w[1,n]|_{a_0}, |w[1,n]|_{a_1}, ..., |w[1,n]|_{a_s})\ (n=0, 1, 2, ...)$$

holds. We say that a word $w\in(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$ is metrically finite/bounded if $|w|_{metr}<\infty$, and metrically infinite/unbounded if $|w|_{metr}=\infty$. We say a word $w\in(\mathcal{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$ is spacially bounded (resp., spacially unbounded) if the set PR(w) is bounded (resp., unbounded) in the unitary space) $\mathbb{C}^{s+1}$. We give some examples of infinite words $w\in(\{a, b\}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$:

(i) $w=a^1b^1a^{\sqrt{-1}/2}b^{\sqrt{-1}/2}a^{(\sqrt{-1}/2)^2}b^{(\sqrt{-1}/2)^2}a^{(\sqrt{-1}/2)^3}b^{(\sqrt{-1}/2)^3}\cdots$ is spacially bounded and metrically bounded with $|w|_{metr}=4<\infty$.

(ii) $w=a^1b^1a^{\sqrt{-1}}b^{\sqrt{-1}}a^{\sqrt{-1}^2}b^{\sqrt{-1}^2}a^{\sqrt{-1}^3}b^{\sqrt{-1}^3}\cdots$ is spacially bounded and metrically unbounded, i.e., $|w|_{metr}=\infty$.

(iii) the Rauzy-Arnoux word w=**abacabaabacababac** $\cdots$ is metrically unbounded and spacially unbounded, cf. the Arnoux-Rauzy substitution ς in Remark 8.

It is clear the following implications:

$$w\text{: spacially unbounded} \Rightarrow |w|_{metr}=\infty \Rightarrow |w|_{symb}=\infty.$$

We shall consider the "Rauzy sets" not only for spacially unbounded words w but also for symbolically infinite words which are spacially bounded or metrically finite, cf. Section 5.

We can consider $(\mathscr{A}^{\wedge}\mathbb{R}^{\times})^{\mathbb{Z}_{>0}}$ as well as $(\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}}$ and define $PR(w) \subset \mathbb{R}^{s+1}$ for $w \in (\mathscr{A}^{\wedge}\mathbb{R}^{\times})^{\mathbb{Z}_{>0}}$. It is natural to define PR(w), and LR(w) as a subset of the Euclidean space $\mathbb{R}^{s+1}$ instead of the unitary space $\mathbb{C}^{s+1}$ for $w \in (\mathscr{A}^{\wedge}\mathbb{R}^{\times})^{\mathbb{Z}_{>0}}$.

§3. Substitution over $\mathscr{A}^{\wedge}\mathbb{C}^{\times}$

For a complex number $z \in \mathbb{C}^{\times}$ and a finite word $w = w_1^{z_1} w_2^{z_2} \cdots w_n^{z_n} \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{*}$ $(w_j \in \mathscr{A},\ z_j \in \mathbb{C}^{\times})$, we define

(1) $$[w]^{z} := w_1^{z \cdot z_1} w_2^{z \cdot z_2} \cdots w_n^{z \cdot z_n}.$$

It is clear that

(2) $$|[w]^{z}|_{a} = z \cdot |w|_{a} \ (\forall w \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{*}, \forall a \in \mathscr{A}, \forall z \in \mathbb{C}^{\times})$$

follows from the definition (1). Let σ be a map

$$\sigma: \mathscr{A} = \{a_0, a_1, \ldots, a_s\} \longrightarrow (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{+} := \bigcup_{1 \le n < \infty} (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\{1,2,\ldots,n\}},$$

(3) $$\sigma(a_j) = b_{1,j}^{z_{1,j}} b_{2,j}^{z_{2,j}} \cdots b_{k_j,j}^{z_{k_j,j}} \ (b_{i,j} \in \mathscr{A},\ z_{i,j} \in \mathbb{C}^{\times}\ (1 \le i \le k_j,\ 0 \le j \le s)).$$

Then it can be extended to a map (which will be also denoted by σ and called a substitution over $\mathscr{A}^{\wedge}\mathbb{C}^{\times}$)

$$(\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{*} \cup (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}} \longrightarrow (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{*} \cup (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}},$$

by

$$\sigma(\lambda) := \lambda,$$

(4) $$\sigma(a_j^{z}) := [\sigma(a_j)]^{z} (= b_{1,j}^{z \cdot z_{1,j}} b_{2,j}^{z \cdot z_{2,j}} \cdots b_{k_j,j}^{z \cdot z_{k_j,j}}),\ 0 \le j \le s,$$

$$\sigma(w) := \sigma(w(1))\sigma(w(2)) \cdots \sigma(w(n)) \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{+}$$

for $w = w(1)w(2) \cdots w(n) \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{+}$, and

$$\sigma(w) := \sigma(w(1))\sigma(w(2)) \cdots \sigma(w(n)) \cdots \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}}$$

for $w = w(1)w(2) \cdots w(n) \cdots \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}}$. The operation $[\ ]^{z}$ will be referred to as a "*local compulsion.*" The definition of the substitution has given by the local compulsion (1); for some other definitions, cf. Section 8. For a substitution σ defined by (3) (together with (1), (4)), we denote by $\mathrm{Exp}(\sigma)$ the set

$$\mathrm{Exp}(\sigma) := \{z_{i,j} \in \mathbb{C}^{\times};\ 1 \le i \le k_j,\ 0 \le j \le s\},$$

which is the set of the numbers $z_{j,k} \in \mathbb{C}^{\times}$ coming from the exponents in (3). We define the incidence matrix M_{σ} of a substitution σ over $\mathscr{A}^{\wedge}\mathbb{C}^{\times}$ by

$$M_{\sigma} := (|\sigma(a_j)|_{a_i})_{0 \le i \le s,\, 0 \le j \le s} \in \mathfrak{m}_{s+1}(\mathbb{C}) \text{ (:=the set of (s+1)×(s+1) matrices of complex entries)}$$

Related to the composition of two substitutions, we have the following

Lemma 1. Let σ and τ be substitutions over $\mathscr{A}^{\wedge}\mathbb{C}^{\times}$. Then $M_{\sigma\tau} = M_{\sigma} M_{\tau}$.

Proof. By (2) we get $|\sigma\tau(a_k)|_{a_i} = |\sigma(\tau(a_k))|_{a_i} = \Sigma_{0 \le j \le s} |\sigma(a_j)|_{a_i} |\tau(a_k)|_{a_j}$. ■

The n-fold iteration of σ is denoted by σ^{n} as usual (by definition σ^{0} is the identity map on $(\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{*} \cup (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}}$. By Lemma 1, we have $M_{\sigma^n} = M_{\sigma}^{n}$, which plays an important rolle related to Rauzy sets for the fixed point of complex powered substitution, cf. Lemma 6, Section 5. We say that symbolically infinite word $w \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}}$ is a fixed point of σ iff $\sigma(w) = w$. Suppose that there exists a symbolically infinite word w that has $\sigma^{n}(a)$ as its prefix for all $n \ge 0$. Such a word w is uniquely determined by σ and a, which is denoted by $\lim \sigma^{n}(a)$.

Lemma 2. Let σ be a substitution over $\mathscr{A}^{\wedge}\mathbb{C}^{\times}$ defined by (3) satisfying

$$\exists a \in \mathscr{A} \text{ such that } \sigma(a) = a^{z}u,\ u \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{+},\ z \in \mathbb{C}^{\times}.$$

Then there exists a unique fixed point $w = \lim \sigma^{n}(a) \in (\mathscr{A}^{\wedge}\mathbb{C}^{\times})^{\mathbb{Z}_{>0}}$ of the substitution prefixed by a iff $z = 1$.

Proof. Suppose $z\neq 1$. Then $\sigma^n(a)$ is prefixed by a^{z^n}, so that σ has no fixed point prefixed by a. Suppose z=1. Then, we see that

$$w:=au\sigma(u)\sigma^2(u)\sigma^3(u)\cdots$$

is an symbolically infinite word that satisfies $\sigma(w)=\sigma(a)\sigma(u)\sigma^2(u)\sigma^3(u)\cdots=au\sigma(u)\sigma^2(u)\sigma^3(u)\cdots=w$, so that w is a fixed point of σ which is prefixed by a. Conversely, any fixed point prefixed by a should have $\sigma^n(a)=au\sigma(u)\sigma^2(u)\sigma^3(u)\cdots\sigma^{n-1}(u)$ as its prefix, which says the uniqueness of the fixed point of the substitution prefixed by a. ■

Remark 3. It is clear that the following two assertions hold: If $\sigma(a)=b^z u$ ($a\neq b$, $z\in\mathbb{C}^\times$), then σ does not have a fixed point prefixed by a. If $\sigma(a)=a$, then w=aaa... is a fixed point of σ prefixed by a, and there are possibly some other fixed points prefixed by a in general. For instance, all the symbolically infinite words are fixed points of σ if σ is the identity map on $(\mathscr{A}^\wedge\mathbb{C}^\times)^* \cup (\mathscr{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$.

We give some examples of substitutions $\tau_1, \ldots, \tau_5$ over $\{a, b\}^\wedge\mathbb{C}^\times$ to understand the definitions already given: Let $\xi, \eta\in\mathbb{C}^\times$.

(i) $\tau_1(a):=aaab^\xi$, $\tau_1(b):=a^\eta$.

$\lim \tau_1{}^n(a)=aaab^\xi aaab^\xi aaab^\xi a^{\xi\eta} aaab^\xi aaab^\xi aaab^\xi a^{\xi\eta} aaab^\xi aaab^\xi aaab^\xi a^{\xi\eta} a^{\xi\eta} a^{\xi\eta} a^{\xi\eta} b^{\xi^2\eta}\cdots$

(ii) $\tau_2(a):=aa^2b^\xi$, $\tau_2(b):=a^\eta$.

$\lim \tau_2{}^n(a)=a\, a^2\, b^\xi\, a^2\, a^4\, b^{2\cdot\xi}\, a^{\xi\eta}\, a^2\, a^4\, b^{2\cdot\xi}\, a^4\, a^8\, b^{4\cdot\xi}\, a^{2\cdot\xi\eta}\, a^{\xi\eta}\, a^{2\cdot\xi\eta}\, b^{\xi^2\eta}\cdots$

(iii) $\tau_3(a):=a^2\, a\, b^\xi$, $\tau_3(b):=a^\eta$.

$\tau_3(a):=a^2 a\, b^\xi$, $\tau_3{}^2(a)=a^4\, a^2\, b^{2\cdot\xi}\, a^2\, a\, b^\xi\, a^{\xi\eta}$, ..., τ_3 has no fixed point.

The incidence matrices M_{τ_j} (j=1, 2, 3) coincide with a matrix $\begin{pmatrix} 3 & \eta \\ \xi & 0 \end{pmatrix}$.

(iv) $\tau_4(a):=a^\xi\, b\, a^{\bar{\xi}}$, $\tau_4(b):=a$ ($\xi=3/5-4\sqrt{-1}/5$, $\bar{\xi}=3/5+4\sqrt{-1}/5$). τ_4 has no fixed point.

τ_4 is not a substitution over $\{a, b\}^\wedge\mathbb{R}^\times$, but $M_{\tau_4}=\begin{pmatrix} 6/5 & 1 \\ 1 & 0 \end{pmatrix}\in\mathfrak{M}_2(\mathbb{R})$.

(v) $\tau_5(a):=a\, b^\zeta$, $\tau_5(b):=a^\zeta$ with $\zeta:=2(-1+\sqrt{-3})/5$ ($|\zeta|<1$).

$\lim \tau_5{}^n(a)=a\, b^\zeta\, a^{\zeta^2} a^{\zeta^2} b^{\zeta^3} a^{\zeta^2} b^{\zeta^3} a^{\zeta^4}\cdots$ is a metrically unbounded word, cf. Example 6, Section 7. If we ignore all the exponent of w, we get the Fibonacci word abaababa $\cdots$.

We can define Rauzy set not only for a substitution having a fixed point, but also for certain substitution having no fixed point like the substitution τ_4 as in (iv) given above. We give a class of such substitutions in the succeeding section. A substitution over $\mathscr{A}^\wedge\mathbb{C}^\times$ will be called as $\mathbb{C}$-substitution. We can consider $\mathbb{R}$-substitutions, which are substitutions over $\mathscr{A}^\wedge\mathbb{R}^\times$.

§4. An equivalence

Let σ be a substitution over $\{a, b\}^\wedge\mathbb{R}^\times$ defined by $\sigma(a)=aab$, $\sigma(b)=a$. Then σ has a unique fixed point $\lim \sigma^n(a)=aab\sigma(ab)\sigma^2(ab)\sigma^3(ab)\cdots$, which coinsides with the fixed point of usual substitution over $\{a, b\}$. The local compulsion makes change the meaning of some fixed points for certain substitutions of usual sense. Recall that the substitution τ_3 given by (iii), Section 3 has no fixed point. Considering such a phenomenon, we introduce an equivalence relation ~ on the set of substitutions.

We put

$$\mathrm{Sub}(\mathscr{A}^\wedge\mathbb{C}^\times):=\{\forall\sigma \text{ substitution over } \mathscr{A}^\wedge\mathbb{C}^\times\}.$$

For $\sigma\in\mathrm{Sub}(\mathscr{A}^\wedge\mathbb{C}^\times)$ defined by (3) and $z\in\mathbb{C}^\times$, we denote by $\sigma^{[z]}$ a substitution defined by

$$\sigma^{[z]}(a_j):=[\sigma(a_j)]^z \ (\forall a_j\in\mathscr{A}).$$

$$(=b_{1,j}{}^{z\cdot z_{1,j}}b_{2,j}{}^{z\cdot z_{2,j}}\cdots b_{k_j,j}{}^{z\cdot z_{k_j,j}}\ (b_{i,j}\in\mathscr{A},\ z_{i,j}\in\mathbb{C}^\times\ (1\le i\le k_j,\ 0\le j\le s)))$$

For σ, $\tau\in\mathrm{Sub}(\mathscr{A}^\wedge\mathbb{C}^\times)$, we say σ is similar to τ and write

$$\sigma\sim\tau$$

if there exists a number $z\in\mathbb{C}^\times$ such that $\sigma=\tau^{[z]}$. In particular, we say σ is congruent to τ and write

$$\sigma\equiv\tau$$

if there exists a number $z\in\mathbb{C}^\times$ such that $\sigma=\tau^{[z]}$ and $|z|=1$. Both "$\sim$" and "$\equiv$" are equivalence relation on $\mathrm{Sub}(\mathscr{A}^\wedge\mathbb{C}^\times)$. For $(a,z)\in\mathscr{A}\times\mathbb{C}^\times$, we define

$$\mathrm{Sub}_{a^z}(\mathscr{A}^\wedge\mathbb{C}^\times):=\{\forall\sigma\in\mathrm{Sub}(\mathscr{A}^\wedge\mathbb{C}^\times);\text{ such that }\sigma(a)=a^z u\ (\exists u\in(\mathscr{A}^\wedge\mathbb{C}^\times)^+)\}.$$

Lemma 4. Let $z\in\mathbb{C}^\times$. Then $\sigma^n(a)=[(\sigma^{[1/z]})^n(a)]^{z^n}$ ($\forall n\in\mathbb{Z}_{\ge0}$, $\forall\sigma\in\mathrm{Sub}_{a^z}(\mathscr{A}^\wedge\mathbb{C}^\times)$).

Proof. For $\sigma\in\mathrm{Sub}(\mathscr{A}^\wedge\mathbb{C}^\times)$, we have

$\sigma(a)=a^z u$,

$\sigma^2(a)=\sigma(a^z u)=\sigma(a^z)\,\sigma(u)=[a^z u]^{z}\,\sigma(u)=a^{z^2}[u]^{z}\,\sigma(u)$,

$\sigma^3(a)=\sigma(a^{z^2}[u]^{z}\,\sigma(u))=\sigma(a^{z^2})\,\sigma([u]^{z})\,\sigma^2(u)=[a^z u]^{z^2}[\sigma(u)]^{z}\,\sigma^2(u)=a^{z^3}[u]^{z^2}[\sigma(u)]^{z}\,\sigma^2(u)$,

$\ldots$,

which together with

$\sigma^{[1/z]}(a)=[\sigma(a)]^{1/z}=[a^z u]^{1/z}=a\,[u]^{1/z}$,

$(\sigma^{[1/z]})^2(a)=\sigma^{[1/z]}(a\,[u]^{1/z})=\sigma^{[1/z]}(a)\,\sigma^{[1/z]}([u]^{1/z})=a\,[u]^{1/z}[\sigma(u)]^{1/z^2}$,

$(\sigma^{[1/z]})^3(a)=\sigma^{[1/z]}(a\,[u]^{1/z}[\sigma(u)]^{1/z^2})$

$=\sigma^{[1/z]}(a)\,\sigma^{[1/z]}([u]^{1/z})\,\sigma^{[1/z]}([\sigma(u)]^{1/z^2})=a\,[u]^{1/z}[\sigma(u)]^{1/z^2}[\sigma^2(u)]^{1/z^3}$,

$\ldots$,

we get inductively $\sigma^n(a)=[(\sigma^{[1/z]})^n(a)]^{z^n}$. ∎

Let $\sigma\in\mathrm{Sub}_{a^z}(\mathscr{A}^\wedge\mathbb{C}^\times)$, $z\in\mathbb{C}^\times$. Then $\sigma(a)=a^z u$ ($u\in(\mathscr{A}^\wedge\mathbb{C}^\times)^+$), so that $\sigma^{[1/z]}(a)=a\,[u]^{1/z}$. The substitution $\sigma^{[1/z]}$ will be referred to as the normalized substitution of σ (with respect to the symbol a). In view of Lemma 2, we may suppose the existence of the fixed point $w\in(\mathscr{A}^\wedge\mathbb{C}^\times)^{\mathbb{Z}_{>0}}$ for the normalized substitution $\sigma^{[1/z]}$ of $\sigma\in\mathrm{Sub}_{a^z}(\mathscr{A}^\wedge\mathbb{C}^\times)$. By Lemma 4, we get

Lemma 5. Let $\sigma\in\mathrm{Sub}_{a^z}(\mathscr{A}^\wedge\mathbb{C}^\times)$, $z\in\mathbb{C}^\times$ and let $\sigma^{[1/z]}$ be the normalized substitution of σ. Then $\mathrm{PR}((\sigma^{[1/z]})^n(a))=z^{-n}\,\mathrm{PR}(\sigma^n(a))$, and $\mathrm{LR}((\sigma^{[1/z]})^n(a))=z^{-n}\,\mathrm{LR}(\sigma^n(a))$.

Lemma 5 says that if σ is similar to τ, then the shape $\mathrm{PR}(\sigma^n(a))$ (resp., $\mathrm{LR}(\sigma^n(a))$) is similar (i.e., the same shape possibly different size) to the shape $\mathrm{PR}(\tau^n(a))$ (resp., $\mathrm{LR}(\tau^n(a))$ in the unitary space $\mathbb{C}^{s+1}$. In particular, if σ is congruent to τ, then $z=e^{i\theta}$ ($-\pi<\exists\theta\le\pi$), so that $\mathrm{PR}(\sigma^n(a))$ and $\mathrm{PR}(\tau^n(a))$ are congruent by the rotation $n\theta$, i.e., $\mathrm{LR}(\sigma^n(a))=e^{in\theta}\cdot\mathrm{LR}((\sigma^{[1/z]})^n(a))$, and so are $\mathrm{LR}(\sigma^n(a))$ and $\mathrm{LR}(\tau^n(a))$.

§5. Convergence and Rauzy set

In this section, we give some results on convergence related to a sequence of powers of an incidence matrix M_σ of a $\mathbb{C}$-substitution σ. We do not need the existence of a fixed point of the substitution σ for some of the convergence results, but for simplicity, we suppose that $\sigma\in\mathrm{Sub}_a(\mathscr{A}^\wedge\mathbb{C}^\times)$, cf. Remark 8. Throughout this section σ denotes a substitution defined by (3), so that $b_{1,0}=a_0=a$ and $z_{1,0}=1$ hold in (3) unless otherwise mentioned. Recall the definition of $\mathrm{Exp}(\sigma)$ in Section 3. It is clear that $1\in\mathrm{Exp}(\sigma)$. We write

$$Z(\sigma):=\mathbb{Z}[\mathrm{Exp}(\sigma)],$$

which is the ring generated by the elements of $\mathrm{Exp}(\sigma)$. We put

$$E=E(\sigma):=\mathbb{Q}(M_\sigma),$$

where the right-hand side denotes a number field generated by the set $\mathbb{Q}$ together with all the components of the matrix M_σ. We set

(5) $$K=K(\sigma):=\mathbb{Q}(\mathrm{Exp}(\sigma)),$$

which is the quotient field of $Z(\sigma)$. We denote by φ_M the characteristic polynomial of a square matrix M, and by $F=F(\sigma)$

$$F=F(\sigma):=\mathbb{Q}(\Phi) \ \ (\Phi:=\text{the set of coefficients of } \varphi_{M_\sigma}),$$

the field generated by Φ. In the following lemma, we suppose that $\varphi_{M_\sigma}\in F[x]\subset E[x]$ is irreducible over E. We denote by $L=L(\sigma)$ the splitting field of φ_{M_σ} over E. It is clear that

$$\mathbb{Z}\subset Z(\sigma)\subset K(\sigma),\ \ \mathbb{Q}\subset F(\sigma)\subset E(\sigma)\subset K(\sigma) \text{ and } \mathbb{Q}\subset F(\sigma)\subset L(\sigma).$$

The field L is algebraic over E, while the fields F, E, K and L are possibly transcendental over $\mathbb{Q}$.

Let $w=w_1^{z_1}w_2^{z_2}w_3^{z_3}\cdots=\lim\sigma^n(a)\in(\mathscr{A}^{\mathbb{C}^\times})^{\mathbb{Z}_{>0}}$ $(w_j\in\mathscr{A},\ z_j\in\mathbb{C}^\times)$ be the fixed point of σ prefixed by a. Recall the

definition of the point-set representation $PR(w):=\{\underline{\underline{p}}(w,1), \underline{\underline{p}}(w,2), \ldots, \underline{\underline{p}}(w,n), \ldots\}$, Section 2. It is clear that

$$\underline{\underline{p}}(w, n)=(|w[1,n]|_{a_0}, |w[1,n]|_{a_1}, \ldots, |w[1,n]|_{a_s})\in Z(\sigma)^{s+1} \ (n=0, 1, 2, \ldots).$$

We define

$$\underline{\pi}(w, n):=1/|w[1,n]|_{a_0}\cdot(|w[1,n]|_{a_1}, \ldots, |w[1,n]|_{a_s})\in K(\sigma)^s$$

for n satisfying $|w[1,n]|_{a_0}\neq 0$. In this section, we consider the convergence of

$$\lim_{n\to\infty}\underline{\pi}(w, |\sigma^n(a)|_{\mathrm{symb}}).$$

Lemma 6. Let σ be a $\mathbb{C}$-substitution, and let $\varphi_{M_\sigma}\in F[x](\subset E[x])$ be irreducible over $E=E(\sigma)$. Suppose that among the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_s$ of $M_\sigma\in\mathfrak{m}_{s+1}(E)$, there exists the dominant eigenvalue $\lambda^\#$ that exceeds the other eigenvalues in modulus , i.e.,

$$|\lambda^\#|>|\lambda_\#|\geq|\lambda_2|\geq\ldots\geq|\lambda_s| \ (\lambda^\#:=\lambda_0,\ \lambda_\#:=\lambda_1).$$

Suppose that there exists an element $a_0\in\mathscr{A}$ such that there exists an integer $n_i>0$ satisfying $|\sigma^{n_i}(a_0)|_{a_i}\neq 0$ for each $0\leq i\leq s$, where n_i is an integer depending on i. Then the limit

(6) $$\lim_{n\to\infty}|\sigma^n(a_0)|_{a_i}/|\sigma^n(a_0)|_{a_0}$$

exists for all $1\leq i\leq s$.

Proof. We consider the set LRS of all linear recurrence sequences over E with φ_{M_σ} as their common characteristic polynomial. By the irreducibility of φ_{M_σ}, $\lambda_j\in\mathbb{C}(0\leq j\leq s)$ are distinct numbers different from zero. Hence, the sequences $\{\lambda_j^n\}_{n=0,1,2,\ldots}$ $(0\leq j\leq s)$ form an E-basis of LRS. By the Cayley-Hamilton theorem, we see that $\{|\sigma^n(a_0)|_{a_i}\}_{n=0,1,2,\ldots}\in\mathrm{LRS}$ $(0\leq\forall i\leq s)$. Therefore we can write

$$|\sigma^n(a_0)|_{a_i}=\gamma_{i0}\lambda_0^n+\gamma_{i1}\lambda_1^n+\cdots+\gamma_{is}\lambda_s^n\in E \ \ (\gamma_{ij}, \lambda_j\in L,\ 0\leq i\leq s,\ 0\leq j\leq s),$$

where L is the splitting field of $\varphi_{M_\sigma}\in E[x]$ over E. Let Gal(L/E) be the Galois group L over E. By the transitivity of the action on L of the Galois group, there exists $g\in\mathrm{Gal}(L/E)$ such that

$$g(\lambda_i)=\lambda_j \text{ for each } 0\leq i\leq s,\ 0\leq j\leq s,$$

and

$$E\ni\gamma_{i0}\lambda_0^n+\gamma_{i1}\lambda_1^n+\cdots+\gamma_{is}\lambda_s^n=|\sigma^n(a_0)|_{a_i}$$
$$=g(|\sigma^n(a_0)|_{a_i})=g(\gamma_{i0}\lambda_0^n+\gamma_{i1}\lambda_1^n+\cdots+\gamma_{is}\lambda_s^n).$$

By the linear independence of the sequences $\{\lambda_j^n\}_{n=0,1,2,\ldots}\in\mathrm{LRS}$ $(0\leq j\leq s)$, γ_{ij} $(0\leq j\leq s)$ are the algebraic conjugates of γ_{i0} over E for each $0\leq i\leq s$. Hence $\gamma_{ik}=0$ $(0\leq\exists i\leq s,\ 0\leq\exists k\leq s)$ implies $\gamma_{ij}=0$ for all $0\leq j\leq s$, which contradicts the hypothesis

$$0\le\forall i\le s\ \exists n_i>0 \text{ such that } |\sigma^{n_i}(a_0)|_{a_i}\neq 0.$$

Therefore we get

$$\gamma_{ij}\neq 0 \text{ for all } 0\le i\le s,\ 0\le j\le s,$$

in particular, $\gamma_{00}\neq 0$. Hence, by setting $\kappa_i:=\lambda_i/\lambda_0$ ($1\le i\le s$), we get

$$\begin{aligned}
&|\sigma^n(a_0)|_{a_i} / |\sigma^n(a_0)|_{a_0} - \gamma_{i0} / \gamma_{00}\\
&=(\gamma_{i0}\lambda_0{}^n+\gamma_{i1}\lambda_1{}^n+\cdots+\gamma_{is}\lambda_s{}^n)/(\gamma_{00}\lambda_0{}^n+\gamma_{01}\lambda_1{}^n+\cdots+\gamma_{0s}\lambda_s{}^n)-\gamma_{i0}/\gamma_{00}\\
&=(\gamma_{i0}+\gamma_{i1}\kappa_1{}^n+\cdots+\gamma_{is}\kappa_s{}^n)/(\gamma_{00}+\gamma_{01}\kappa_1{}^n+\cdots+\gamma_{0s}\kappa_s{}^n)-\gamma_{i0}/\gamma_{00}\\
&=(\gamma_{00}(\gamma_{i1}\kappa_1{}^n+\cdots+\gamma_{is}\kappa_s{}^n)-\gamma_{i0}(\gamma_{01}\kappa_1{}^n+\cdots+\gamma_{0s}\kappa_s{}^n))/(\gamma_{00}(\gamma_{00}+\gamma_{01}\kappa_1{}^n+\cdots+\gamma_{0s}\kappa_s{}^n))\\
&=((\gamma_{00}\gamma_{i1}-\gamma_{i0}\gamma_{01})\kappa_1{}^n+\cdots+(\gamma_{00}\gamma_{is}-\gamma_{i0}\gamma_{0s})\kappa_s{}^n)/(\gamma_{00}(\gamma_{00}+\gamma_{01}\kappa_1{}^n+\cdots+\gamma_{0s}\kappa_s{}^n))\\
(7)\quad &=O(\kappa_1{}^n)\to 0 \text{ (as } n\to\infty),
\end{aligned}$$

since $1>|\kappa_1|=|\lambda_\# / \lambda^\#|\ge|\lambda_i / \lambda_0|\ge|\kappa_i|$ ($i=1, \ldots, s$). Therefore we get

$$\lim_{n\to\infty} |\sigma^n(a_0)|_{a_i} / |\sigma^n(a_0)|_{a_0} = \gamma_{i0} / \gamma_{00}\ (0\le i\le s). \blacksquare$$

A substitution σ satisfying all the condition stated in Lemma 6 will be referred to as a *dominant substitution*, and an eigenvalue (including $\lambda_\#$ itself) having the same modulus as $\lambda_\#$ will be referred to as a *subdominant eigenvalue*.

Theorem 7. Let σ be a dominant substitution. Then

$$\lim_{n\to\infty} \underline{\pi}(w, |\sigma^n(a)|_{symb})=(\xi_1, \ldots, \xi_s),\ \xi_i\neq 0\ (1\le\forall i\le s),$$

where w is the fixed point of σ prefixed by a, and $\underline{\underline{\xi}}= {}^t(1, \xi_1, \ldots, \xi_s)$ is the eigenvector with respect the dominant eigenvalue of M_σ.

Proof. Let $\lambda_0, \lambda_1, \ldots, \lambda_s$ be the eigenvalues of M_σ as in Lemma 6. Let $\underline{\underline{\xi}}=\underline{\underline{\xi_0}}, \underline{\underline{\xi_1}}, \ldots, \underline{\underline{\xi_s}}$ be the column eigenvectors with respect to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_s$ of M_σ respectively. The numbers $\lambda_0, \lambda_1, \ldots, \lambda_s$ are distinct by the irreducibility of φ_{M_σ}, so that the vectors $\underline{\underline{\xi_i}}$ ($0\le i\le s$) are linearly independent over $E(\sigma)$. We put

$$\Xi:=(\underline{\underline{\xi_0}}, \underline{\underline{\xi_1}}, \ldots, \underline{\underline{\xi_s}}),\ D:=D[\lambda_0, \lambda_1, \ldots, \lambda_s],$$

where $D[\lambda_0, \lambda_1, \ldots, \lambda_s]$ denotes the diagonal matrix with diagonal entries $\lambda_0, \lambda_1, \ldots, \lambda_s$. Then Ξ is non-singular and

$$M_{\sigma^n}=M_\sigma{}^n = \Xi\, D^n\, \Xi^{-1} =(\lambda_0{}^n\underline{\underline{\xi_0}}, \lambda_1{}^n\underline{\underline{\xi_1}} \ldots \lambda_s{}^n\underline{\underline{\xi_s}})\, \Xi^{-1}.$$

Hence

$${}^t(|\sigma^n(a_0)|_{a_0}, |\sigma^n(a_0)|_{a_1}, \ldots, |\sigma^n(a_0)|_{a_s})=\eta_0\lambda_0{}^n\underline{\underline{\xi_0}}+\eta_1\lambda_1{}^n\underline{\underline{\xi_1}}+\cdots+\eta_s\lambda_s{}^n\underline{\underline{\xi_s}}$$

holds for some constants $\eta_i\in\mathbb{C}$. By the proof of Lemma 6, we have $0\neq\gamma_{i0}=\eta_0\xi_j\neq 0$ for all $0\le i\le s$, where ${}^t(\xi_0, \xi_1, \ldots, \xi_s)=\underline{\underline{\xi_0}}$. Hence, by replacing $\underline{\underline{\xi_0}}$ by $\xi_0{}^{-1}\underline{\underline{\xi_0}}$ if necessary, we may suppose, $\underline{\underline{\xi_0}}= {}^t(1, \xi_1, \ldots, \xi_s)$. We get

$$\begin{aligned}
&\lim_{n\to\infty} 1/(\eta_0\lambda_0{}^n)\cdot {}^t(|\sigma^n(a_0)|_{a_0}, |\sigma^n(a_0)|_{a_1}, \ldots, |\sigma^n(a_0)|_{a_s})\\
&=\lim_{n\to\infty} 1/(\eta_0\lambda_0{}^n)\cdot(\eta_0\lambda_0{}^n\underline{\underline{\xi_0}}+\eta_1\lambda_1{}^n\underline{\underline{\xi_1}}+\cdots+\eta_s\lambda_s{}^n\underline{\underline{\xi_s}})\\
&=\underline{\underline{\xi_0}}= {}^t(1, \xi_1, \ldots, \xi_s),
\end{aligned}$$

which implies the lemma. $\blacksquare$

In what follows, we only consider dominant substitutions unless otherwise mentioned. For the numbers ξ_j ($1\le i\le s$) appeared in the proof of Theoem 7, we set

$$\underline{\delta}(w)=\underline{\delta}(\sigma; a_0):=(1, \xi_1, \ldots, \xi_s) \in L(\sigma)^{s+1} \subset \mathbb{C}^{s+1},$$

which will be called as the *direction* of the fixed point $w=w(\sigma, a_0)=\lim \sigma^n(a_0)$. We put

$$\underline{\delta}(w):=(\xi_1, \ldots, \xi_s)=\lim_{n\to\infty} \underline{\pi}(w, |\sigma^n(a_0)|_{\mathrm{symb}}) \in L(\sigma)^s \subset \mathbb{C}^s,$$

and we call the vector $\underline{\delta}(w)$ the *projective direction* of w. The vector $\underline{\pi}(w,n)=\underline{\pi}(w(\sigma, a_0), n)$ will be referred to as *substitution fractions* for $\underline{\delta}(w)$, or simply as *intermediate convergents.* The vector $\underline{\pi}(w, |\sigma^n(a_0)|_{\mathrm{symb}})$ will be called as *principal convergents* (of substitution fractions of $\underline{\delta}(w(\sigma, a_0))$). An intermediate convergent $\underline{\pi}(w, n)$ gives a simultaneous approximation of

$$\underline{\delta}(w) \in L(\sigma)^s \subset \mathbb{C}^s$$

by the fractions

$$|w[1,n]|_{a_i} / |w[1,n]|_{a_0} \in K(\sigma) \quad (1\le i\le s)$$

with a common denominator $|w[1,n]|_{a_0}\in Z(\sigma)$ and numerators $|w[1,n]|_{a_i}\in Z(\sigma)$ $(1\le i\le s)$. If $|\lambda^\#|\neq 1$ and $|w[1,n]|_{a_0}\neq 0$, then we can define $\mu^{(i)}(n)$ by an equality

(8) $$\left| \xi_i - |w[1,n]|_{a_i} / |w[1,n]|_{a_0} \right| = \left| |w[1,n]|_{a_0} \right|^{-\mathrm{sgn}(\log|\lambda^\#|)\cdot \mu^{(i)}(n)} \quad (1\le i\le s)$$

for n satisfying $|w[1,n]|_{a_0} \neq 0$, where sgn is the sign function, i.e., sgn(x)=1 (x>0), sgn(x)=-1 (x<0). The quantity $\mu_n^{(i)}$ measures the quality of the simultaneous approximation of $\underline{\delta}(w)$ by $\underline{\pi}(w, n)$. By the proof of Lemma 6, one can show that $|\sigma^n(a_0)|_{a_0}\neq 0$ for all sufficiently large n, so that $\mu^{(i)}(|\sigma^n(a_0)|_{\mathrm{symb}})$ is well-defined for all $n\ge n_0(\sigma)$, where $n_0(\sigma):=1+\mathrm{Max}\{n\in\mathbb{Z}_{>0};\ |\sigma^n(a_0)|_{a_0}=0\}$. For simplicity, we put

$$\nu^{(i)}(n) := \mu^{(i)}(|\sigma^n(a_0)|_{\mathrm{symb}}) \quad (n\ge n_0(\sigma)).$$

Remark 8. Theorem 7 says that

(a) the projective direction of the first law vector of the matrix $M_\sigma{}^n$ converges to $(\xi_1, \ldots, \xi_s)$, where ${}^t(1, \xi_1, \ldots, \xi_s)$ is the eigenvector with respect the dominant eigenvalue of M_σ,

and

(b) $(\xi_1, \ldots, \xi_s)$ coincides with the projective direction $\underline{\delta}(w)$ of the fixed point w of the substitution σ prefixed by a_0.

Notice that the assertion (a) completely comes from the matrix M_σ, and independent of the existance of the fixed point of the substitution σ. Introducing the projective direction $\underline{d}(\sigma; k)$ of the (k+1)-st law vector of the matrix $M_\sigma{}^n$ as in the following, the assertion (a) can be extended in twofold senses:

(i) for substitutions σ not necessarily having fixed points,

and

(ii) the projective direction of the (k+1)-st column vector of the matrix $M_\sigma{}^n$ converges to $(\xi_1, \ldots, \xi_s)$ for some $0\le k\le s$.

Let σ be a substitution over $\mathscr{A}$^$\mathbb{C}^\times$ with $\mathscr{A}=\{a_0, a_1, \ldots, a_s\}$. Let $\varphi_{M_\sigma}\in F[x]$ be irreducible over E=E(σ). Suppose that among the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_s$ of $M_\sigma\in \mathfrak{M}_{s+1}(E)$, there exists the dominant eigenvalue $\lambda^\#$ that exceeds the other eigenvalues in modulus. Let $m_{i,j}(n)$ $(0\le i\le s, 0\le j\le s, n>0)$ be complex numbers defined by $(m_{i,j}(n))_{0\le i\le s, 0\le j\le s} := M_\sigma{}^n$. Suppose that there exists a nonempty subset $\mathscr{B}$ of $\mathscr{A}$ such that there exists an integer $n(i,k)>0$ satisfying $|\sigma^{n(i,k)}(a_k)|_{a_i}\neq 0$ for each $0\le i\le s$ and k with $a_k\in\mathscr{B}$. Then we can show the existance of the limit

$$\underline{d}(\sigma; k):=\lim_{n\to\infty} 1/m_{0,k}(n) \cdot (m_{1,k}(n),\ldots,m_{s,k}(n)) \text{ for all } k \text{ with } a_k\in\mathscr{B}.$$

One can show this fact by the same manner as the proof of Lemma 6. Thus one can see that

$$\underline{d}(\sigma; k)=(\xi_1, \ldots, \xi_s)\in L(\sigma)^s \text{ for all } k \text{ with } a_k\in\mathscr{B},$$

where $\underline{\underline{\xi}}= {}^t(1, \xi_1, \ldots, \xi_s)$ is the eigenvector with respect the dominant eigenvalue $\lambda^\#$ of M_σ as in Theorem 7

This fact extends not only Theorem 7 for $\mathbb{C}$-substitutions σ having no fixed point, but also the definition of Rauzy fractals for $\mathbb{C}$-substitutions σ having no fixed point. For example, let σ be a substitution over $\{a_0, a_1, a_2\}$ defined by $\sigma(a_0):=a_1a_0$, $\sigma(a_1):=a_0a_2$, $\sigma(a_2):=a_0$. Then σ has no fixed points, and $\underline{d}(\sigma; k)=(1/\alpha,1/\alpha^2)$ (k=0,1,2;

$\alpha^3+\alpha^2+\alpha+1=0$ $\alpha>1$); the vectors $\underline{d}(\sigma;\, k)$ $(k=0,1,2)$ coincide with the projective direction $\underline{\delta}(w)$ of the fixed point $w=\lim \varsigma^n(a_0)$ of the Arnoux-Rauzy substitution ς defined by $\varsigma(a_0):=a_0a_1$, $\varsigma(a_1):=a_0a_2$, $\varsigma(a_2):=a_0$. We do not go further related to substitutions having no fixed points in this paper, but it will be very interesting to consider Rauzy fractals for substitutions having no fixed points.

Theorem 9. Let λ_i be as in Lemma 6 with $|\lambda^\#|\neq 1$. Let ρ be a positive number defined by

$$\rho := \frac{\log|\lambda^\#/\lambda_\#|}{\mathrm{sgn}\,(\log|\lambda^\#|)\cdot\log|\lambda^\#|} = \frac{\log|\lambda^\#/\lambda_\#|}{|\log|\lambda^\#||}$$

Then

(i) $\liminf_{n\to\infty} \nu^{(i)}(n) \geq \rho$ $(1\leq\forall i\leq s)$.

(ii) $\rho \leq \dfrac{\log\left(|\lambda^\#|^{1+1/(s-1)}/|N_{L/E}(\lambda^\#)|^{1/(s-1)}\right)}{|\log|\lambda^\#||}=\mathrm{sgn}(\log|\lambda^\#|)\cdot(1+\frac{1}{s-1}\cdot(1-\log|N_{L/E}(\lambda^\#)|/\log|\lambda^\#|))$ for $s>1$;

$$\rho=\frac{2\log|\lambda^\#|-\log|N_{L/E}(\lambda^\#)|}{|\log|\lambda^\#||} \text{ for } s=1,$$

where $N_{L/E}(\lambda^\#)$ $(=\varphi_{M_\sigma}(0)=\det M_\sigma)$ is the norm of $\lambda^\#\in L$ over E.

Proof of (i). By the definition of the ρ we have

$$(1<)|\lambda^\#/\lambda_\#|=|\lambda^\#|^{\mathrm{sgn}\,(\log|\lambda^\#|)\cdot\rho}. \tag{9}$$

By (7) we see that there exist a constant $\kappa>0$ independent of n such that

$$\left|\,\xi_i-|\sigma^n(a_0)|_{a_i}/|\sigma^n(a_0)|_{a_0}\right|\leq\kappa\cdot|\kappa_1|^n\ (|\kappa_1|=|\lambda_\#/\lambda^\#|<1),$$

which together with (9) implies

$$\begin{aligned}\nu^{(i)}(n)&=\mathrm{sgn}(\log|\lambda^\#|)\log\left(1/\left|\,\delta_i-|\sigma^n(a_0)|_{a_i}/|\sigma^n(a_0)|_{a_0}\right|\right)/\log\left||\sigma^n(a_0)|_{a_0}\right|\\ &\geq\mathrm{sgn}(\log|\lambda^\#|)\log(1/(\kappa\cdot|\lambda^\#/\lambda_\#|^n))/\log\left||\sigma^n(a_0)|_{a_0}\right|\\ &=\mathrm{sgn}(\log|\lambda^\#|)\log\left(1/\kappa\cdot|\lambda^\#|^{\mathrm{sgn}\,(\log|\lambda^\#|)\cdot\rho n}\right)/\log\left(|\gamma_{00}|\cdot|\lambda^\#|^n\cdot(1+o(1))\right)\\ &=\mathrm{sgn}(\log|\lambda^\#|)\cdot\frac{-\log\kappa+\mathrm{sgn}\,(\log|\lambda^\#|)\,\rho\, n\log|\lambda^\#|}{\log|\gamma_{00}|+n\log|\lambda^\#|+\log|1+o(1)|},\end{aligned}$$

where the o-constant does not depend on n. Since $\log|\lambda^\#|\neq 0$, we get

$$\liminf_{n\to\infty}\nu^{(i)}(n)\geq\mathrm{sgn}(\log|\lambda^\#|)\times\lim_{n\to\infty}\frac{-\log\kappa+\mathrm{sgn}\,(\log|\lambda^\#|)\,\rho\cdot n\cdot\log|\lambda^\#|}{\log|\gamma_{00}|+n\cdot\log|\lambda^\#|+\log|1+o(1)|}=\rho.\ \blacksquare$$

Proof of (ii). Suppose $s>1$. Since $|N_{L/E}(\lambda^\#)|\leq|\lambda^\#||\lambda_\#|^{s-1}$, we have $1\leq|\lambda^\#|^{1/(s-1)}|\lambda_\#|/|N_{L/E}(\lambda^\#)|^{1/(s-1)}$, so that

$$|\lambda^\#/\lambda_\#|\leq|\lambda^\#/\lambda_\#|\cdot(|\lambda^\#|^{1/(s-1)}|\lambda_\#|/|N_{L/E}(\lambda^\#)|^{1/(s-1)})=|\lambda^\#|^{1+1/(s-1)}/|N_{L/E}(\lambda^\#)|^{1/(s-1)},$$

which implies

$$\rho=\frac{\log|\lambda^\#/\lambda_\#|}{|\log|\lambda^\#||}\leq\frac{\log\left(|\lambda^\#|^{1+1/(s-1)}/|N_{L/E}(\lambda^\#)|^{1/(s-1)}\right)}{|\log|\lambda^\#||}.$$

Suppose $s=1$. Since $|\lambda_\#|=|N_{L/E}(\lambda^\#)|/|\lambda^\#|$ for $s=1$, the assertion follows immediately from the definition of ρ. $\blacksquare$

Remark 10. (i) The number $\rho(>0)$ is a measure of the gap between the dominant eigenvalue $\lambda^\#$ and a subdominant eigenvalue $\lambda_\#$.

(ii) $\lim_{n\to\infty}\left||\sigma^n(a_0)|_{a_0}\right|=\infty$ if $\mathrm{sgn}(\log|\lambda^\#|)=+1$, and $\lim_{n\to\infty}|\sigma^n(a_0)|_{a_0}=0$ if $\mathrm{sgn}(\log|\lambda^\#|)=-1$, so that the principal convergent $\underline{\pi}(w,\, |\sigma^n(a_0)|_{\mathrm{symb}})$ always converges if $|\lambda^\#|\neq 1$.

(iii) If σ is a dominant substitution with $|\lambda^\#|>1$ for the dominant eigenvalue of M_σ then its fixed point is spacially unbounded. If M_σ has dominant $|\lambda^\#|<1$, then $\lim_{n\to\infty}|\sigma^n(a_0)|_{a_i}=0$ for all $0\leq i\leq s$.

(iv) It has been known that if $f\in\mathbb{Q}[x]$ is an irreducible polynomial over $\mathbb{Q}$ with roots λ_i ($1\le i\le d=\deg f$) satisfying $|\lambda_1|>|\lambda_2|=|\lambda_3|=\ldots=|\lambda_d|$, then $d\le 3$. Hence, if $K=\mathbb{Q}$ and $d\ge 4$, then the inequality "≤" in the assertion (ii) in Theorem 9 can be replaced by "<", cf. [M].

The convergence of (6) is said to be strong (resp., weak) if

$$\liminf_{n\to\infty} \nu^{(i)}(n)\ (=\liminf_{n\to\infty} \mu^{(i)}(|\sigma^n(a_0)|_{\mathrm{symb}}))>1\quad (1\le\forall i\le s)$$

$$(\text{resp., } \liminf_{n\to\infty} \nu^{(i)}(n)>0\ (1\le\forall i\le s)\ \&\ \liminf_{n\to\infty} \nu^{(i)}(n)\le 1\quad (1\le\exists i\le s)).$$

Let σ be a dominant substitution. We say that $\varphi_{M_\sigma}\in F[x]$ is *#-Diophantine* if

$$|\lambda^\#|>1>|\lambda_\#|$$

holds for dominant and subdominant eigenvalues of φ_{M_σ}. We say that $\varphi_{M_\sigma}\in F[x]$ is *♭-Diophantine* if

$$1>|\lambda^\#|^2>|\lambda_\#|$$

holds. We simply say that $M_\sigma\in F[x]$ is *Diophantine* if it is #-Diophantine, or ♭-Diophantine. (We say that a substitution σ is Diophantine if φ_{M_σ} is Diophantine. We also say that a matrix $M_\sigma\in\mathfrak{m}_{s+1}(E)$ is Diophantine if σ is Diophantine.) It is clear that $\lambda^\#$ is a Pisot number if $\lambda^\#>0$, $\mathrm{Exp}(\sigma)\subset\mathbb{Z}$ (i.e., $Z(\sigma)=\mathbb{Z}$ and $F=\mathbb{Q}$) and M_σ is #-Diophantine.

Theorem 11. Let σ be a Diophantine substitution. Then the principal convergent $\underline{\pi}(w, |\sigma^n(a_0)|_{\mathrm{symb}})$ strongly converges.

Proof. By (i), Theorem 9, we have

$$\liminf_{n\to\infty} \nu^{(i)}(n) \ge\rho=\frac{\log|\lambda^\#/\lambda_\#|}{|\log|\lambda^\#||}.$$

If $|\lambda^\#|>1>|\lambda_\#|$, then $\rho=1+\frac{-\log|\lambda_\#|}{\log|\lambda^\#|}>1$. If $1>|\lambda^\#|^2>|\lambda_\#|$, then $|\lambda^\#/\lambda_\#|>1/|\lambda^\#|$ and $1>|\lambda^\#|$. Hence we obtain

$$\rho=\frac{\log|\lambda^\#/\lambda_\#|}{|\log|\lambda^\#||}>\frac{\log|1/\lambda^\#|}{|\log|\lambda^\#||}=\frac{\log|1/\lambda^\#|}{-\log|\lambda^\#|}=1,$$

Hence, we get the theorem. ■

Remark 12. If $Z(\sigma)$ consists only of algebraic integers, then

$$\underline{\underline{p}}(w, n)=(|w[1,n]|_{a_0}, |w[1,n]|_{a_1}, \ldots, |w[1,n]|_{a_s})\in \mathrm{Ord}(K(\sigma))^{s+1}\ (n=0, 1, 2, \ldots),$$

where $\mathrm{Ord}(K(\sigma))$ is the ring of algebraic integers in $K(\sigma)$. Thus, in such a case, it will be more interesting to consider $\mu_H{}^{(i)}(n)$ defined by

$$\left|\ \xi_i - |w[1,n]|_{a_i} / |w[1,n]|_{a_0}\right| = H_{K/\mathbb{Q}}(\alpha)^{-\mathrm{sgn}(\log|\lambda^\#|)\cdot\mu_H{}^{(i)}(n)}\quad (1\le i\le s)$$

for n satisfying $|w[1,n]|_{a_0}\neq 0$ instead of $\mu^{(i)}(n)$ defined by (8), where $\alpha=|w[1,n]|_{a_0}\in\mathrm{Ord}(K)\subset K$ and $H_{K/\mathbb{Q}}(\alpha)$ denotes the height of a number α over $\mathbb{Q}$, i.e., the maximum value of the absolute value of the coefficients of the minimal polynomial $\in\mathbb{Z}[x]$ of $\alpha\in K$, cf. (i-iii), Fig. 8, Example 1, Section 7. It will be interesting also to consider $\mu_H{}^{(i)}$ for the height $H_{K/\mathbb{Q}}(\alpha)$ with $\alpha=|w[1,n]|_{a_i} / |w[1,n]|_{a_0}\in K$ instead of $\alpha=|w[1,n]|_{a_0}\in\mathrm{Ord}(K)\subset K$.

The so called Rauzy fractal was introduced by Rauzy as a realization of the substitution dynamical system for a given substitution, cf. [Ra]. We now extend the definition of the Rauzy fractal(s) for certain $\mathbb{C}$-substitutions. Recall the notation $w=w(1)w(2)w(3)\cdots$, PR(w), $(\underline{\underline{p}}(w, m-1),\underline{\underline{p}}(w, m)]$, etc. in Section 2. Let $\sigma\in\mathrm{Sub}_{a^z}(\mathscr{A}^{\mathbb{C}^\times})$ be a substitution. Let w be the fixed point of the normalized substitution $\sigma^*=\sigma^{[1/z]}$ of σ prefixed by a. It is clear that

$$\mathrm{PR}(w)=\{\underline{\underline{p}}(w, 1), \underline{\underline{p}}(w, 2), \ldots, \underline{\underline{p}}(w, n), \ldots\}=\mathrm{PR}(\lim(\sigma^*)^n(a))=\lim_{n\to\infty}\mathrm{PR}((\sigma^*)^n(a))$$

holds, since $\{\mathrm{PR}((\sigma^*)^n(a))\}_{n=0,1,2,\ldots}$ is a monotone increasing sequence of sets. Suppose that there exists the direction $\underline{\delta}(w)$. Then we can define the Rauzy set R(w) by

$$R(w)=R(w(\sigma, a)) :=(\mathrm{proj}_{\underline{\delta}(w)}(PR(w)))^{cl} \subset \Pi(\underline{\delta}(w)) \subset \mathbb{C}^{s+1},$$

where $\Pi(\underline{\gamma})$ is the hyperplane $\subset \mathbb{C}^{s+1}$ of dimension s (over $\mathbb{C}$) perpendicular to a vector $\underline{0}\neq\underline{\gamma}\in\mathbb{C}^{s+1}$, and

$$\mathrm{proj}_{\underline{\gamma}} : \mathbb{C}^{s+1} \to \Pi(\underline{\gamma})$$

is the projection to $\Pi(\underline{\gamma})$ along the vector $\underline{\gamma}$, and S^{cl} is the closure of a set $S\subset\mathbb{C}^{s+1}$ with respect to the metric of the unitary space $\mathbb{C}^{s+1}$. To be precise, the Hermitian product $(\underline{u}, \underline{v})$ is not commutative, so that we have some possibilities of the image of $\mathrm{proj}_{\underline{\gamma}}$, but the resulting sets R(w) have not big difference, i.e., one can be the mirror image of the other. The set R(w) has partitions

$$R(w) = \bigcup_{0\le i\le s} R(w; a_i; *)$$

with

$$R(w; a_i, *):=(\mathrm{proj}_{\underline{\delta}(w)}(\{\underline{p}(w, n); w(n)\in a_i{}^\wedge\mathbb{C}^\times, n\in\mathbb{Z}_{>0}\}))^{cl}, 0\le i\le s.$$

We define

$$R(w; *, x):=(\mathrm{proj}_{\underline{\delta}(w)}(\{\underline{p}(w, n); \text{the length of } (\underline{p}(w, n-1), \underline{p}(w, n)]=x, n\in\mathbb{Z}_{>0}\}))^{cl},$$

$$R(w; a_i, x):=(\mathrm{proj}_{\underline{\delta}(w)}(\{\underline{p}(w, n); w(n)\in a_i{}^\wedge\mathbb{C}^\times \ \&\ \text{the length of } (\underline{p}(w, n-1),\underline{p}(w, n)]=x, n\in\mathbb{Z}_{>0}\}))^{cl}.$$

The set $R(w; a_i, *)$ (resp., $R(w; *, x)$, $R(w; a_i, x)$) will be called as the $(a_i, *)$-part (resp., $(*, x)$-part, (a_i, x)-part) of the Rauzy set of w. We can consider Rauzy set not only for σ whose fixed point w is a spacially unbounded word, but also for symbolically infinite word w which is spacially bounded or metrically finite (i.e., $|w|_{metr}<\infty$). By Lemma 6, we see that for an arbitrarily given dominant substitution $\mathrm{Sub}_{a^z}(\mathscr{A}{}^\wedge\mathbb{C}^\times)$, the Rauzy set can be defind. In particular, if $|\lambda^\#|<1$ for the dominant eigenvalue, then the fixed point w is possibly spacially bounded having its direction $\underline{\delta}(w)$, and $\underline{p}(w, |\sigma^n(a)|_{symb})$ converges to $\underline{0}$. We may think that PR(w) itself is the "Rauzy set" for a spacially bounded word $w=\sigma^n(a)$, and we may study the shape

$$\mathrm{proj}_{\underline{\gamma}}(PR(w))\ (\underline{\gamma}\in\mathbb{C}^{s+1}, \underline{\gamma}\neq\underline{0})$$

in connection with simultaneous approximation of the projective direction $\underline{\delta}(w)$ by intermediate covergents, see Examples 2, 3, 6, 7, 9 in Section 7. We shall have a look of a broken line $\mathrm{proj}_{\underline{\gamma}}(PR(\sigma^n(a))$, or $\mathrm{proj}_{\underline{\gamma}}(LR(\sigma^n(a))$ for $\underline{\gamma}$ independent of the direction $\underline{\delta}(\sigma; a)$, since it has interesting/curious shape in some cases.

For an $\mathbb{R}$-substitution $\sigma\in\mathrm{Sub}_{a^z}(\mathscr{A}{}^\wedge \mathbb{R}^\times)$, the set R(w) (as well as $R(w; a_i, *)$, $R(w; *, x)$, $R(w; a_i, x)$ can be defined as a subset of the Euclidean space $\mathbb{R}^{s+1}$.

We denote by Diam(S) the diameter of a set $S\subset\mathbb{C}^{s+1}$ defined by

$$\mathrm{Diam}(S):=\mathrm{Sup}\ \{d(\underline{u}, \underline{v}); \underline{u}, \underline{v}\in S\}\in\mathbb{R}_{>0}\cup\{\infty\}.$$

Usually, the Rauzy set has been considered in the case where

$$\mathrm{Diam}(\mathrm{proj}_{\underline{\delta}(w)}(PR(\sigma^n(a)))\ (n=0, 1, 2, ...)$$

is a bounded sequence, so that the resulting set R(w) becomes a compact set, cf. Conjectures 16-18 at the end of Section 7. We have tried some experiments for dominant substitutions σ which are not diophantine. It seems very likely that there exists a suitable number/normalizer $\kappa(\sigma, a, \underline{\gamma})\in\mathbb{C}^\times$ such that

$$\Big(\bigcap_{0\le n<\infty}\bigcup_{n\le m<\infty} \kappa^m \cdot \mathrm{proj}_{\underline{\gamma}}(PR(\sigma^m(a)))\Big)^{cl}$$

becomes a compact set$\neq\{\underline{0}\}$ even for some substitutions σ which are not diophantine, cf. Fig. 29-33, Example 9 in Section 7.

§6. Continued fractions and substitutions

Let $\underline{\gamma}\in\mathbb{C}^s$ be a column vector, and T, $C(\underline{\gamma})$ be matrices defined by

$$C(\underline{\gamma}) := \begin{pmatrix} 0 & 1 \\ E_s & \underline{\gamma} \end{pmatrix} \in \mathfrak{M}_{s+1}(\mathbb{C}),\ T := (t_{ij})_{0\le i\le s,\, 0\le j\le s},$$

where E_s is the unit matrix of size s×s, and t_{ij}:=1 (if i+j=s), t_{ij}:=0 (otherwise). We denote by ${}^{\times}X := TXT = T^{-1}XT$ the cross transposed matrix of X, and by ${}^{t}X$ the usual transposed matrix of X. If the cross transposed matrix of $M_\sigma := (|\sigma(a_j)|_{a_i})_{0\le i\le s,\, 0\le j\le s} \in \mathfrak{M}_{s+1}(\mathbb{C})$ is a product of matrices of the form

$${}^{\times}M_\sigma = {}^{t}C(\underline{\gamma_1})\ {}^{t}C(\underline{\gamma_2}) \cdots {}^{t}C(\underline{\gamma_k}),$$

and σ has a fixed point w, then the projective direction $\underline{\delta}$(w) of w relates with the "transposed value" (defined below) of an s-dimensional periodic continued fraction

$$CF = [\underline{\gamma_k}; \underline{\gamma_{k-1}}, \ldots, \underline{\gamma_1}, \underline{\gamma_k}, \underline{\gamma_{k-1}}, \ldots, \underline{\gamma_1}, \ldots, \underline{\gamma_k}, \underline{\gamma_{k-1}}, \ldots, \underline{\gamma_1}, \ldots]$$

concerning the notation of CF and the proof of Lemma 13 below, see for example [T1], [FISTY].

Lemma 13. Let s≥2 be an integer and $\gamma_1, \ldots, \gamma_{s-1}$ be given complex numbers. Let $\sigma \in \mathbf{Sub}_{a_0^{\gamma_0}}(\mathscr{A}^{\mathbb{C}^\times})$ be a substitution defined by

$$\sigma(a_i) := a_0^{\gamma_i} a_{i+1}\ (\sigma(a_i) := a_{i+1} \text{ if } \gamma_i = 0) \text{ for } 0\le i\le s-1, \text{ and } \sigma(a_s) := a_0.$$

Then

$${}^{\times}M_\sigma = {}^{t}C(\underline{\gamma}),$$

where $\underline{\gamma} = {}^{t}(\gamma_{s-1}, \gamma_{s-2}, \ldots, \gamma_0) = {}^{t}(|\sigma(a_{s-1})|_{a_0}, |\sigma(a_{s-2})|_{a_0}, \ldots, |\sigma(a_0)|_{a_0})$.

Proof. It is clear by the definition of σ. ■

Lemma 14. Let $\underline{\gamma_n} \in \mathbb{C}^s$ (n=0,1,2, ...). Let Q_n be a matrix defined by

$$Q_n := \begin{pmatrix} q_{n-s}^{(0)} & \cdots & q_n^{(0)} \\ \vdots & \cdots & \vdots \\ q_{n-s}^{(s)} & \cdots & q_n^{(s)} \end{pmatrix} := C(\underline{\gamma_0})C(\underline{\gamma_1}) \cdots C(\underline{\gamma_n}) \in \mathfrak{M}_{s+1}(\mathbb{C}).$$

Then

(10) $$[\underline{\gamma_0}; \underline{\gamma_1}, \ldots, \underline{\gamma_n}] = 1/q_n^{(0)} \cdot {}^{t}(q_n^{(1)}, \ldots, q_n^{(s)}) \quad (n=0, 1, 2, \ldots).$$

The vector $1/q_n^{(0)} \cdot {}^{t}(q_n^{(1)}, \ldots, q_n^{(s)})$ in Lemma 14 is called as the nth convergent of the continued fraction. We do not use Lemma 14 in this paper; but in compare with (10), we define the transposed value ${}^{TV}CF$ of a continued fraction CF= $[\underline{\gamma_0}; \underline{\gamma_1}, \ldots, \underline{\gamma_n}, \ldots]$ by

$${}^{TV}[\underline{\gamma_0}; \underline{\gamma_1}, \ldots, \underline{\gamma_n}] := 1/q_n^{(s)} \cdot {}^{t}(q_{n-1}^{(s)}, \ldots, q_{n-s}^{(s)}),$$

and

$${}^{TV}CF = {}^{TV}[\underline{\gamma_0}; \underline{\gamma_1}, \ldots, \underline{\gamma_n}, \ldots] := \lim_{n\to\infty} {}^{TV}[\underline{\gamma_0}; \underline{\gamma_1}, \ldots, \underline{\gamma_n}],$$

as far as the continued fractions we are concerned are not "*sinnloss*(meaningless)," and the limit exists, where $q_n^{(i)}$ are as in Lemma 14, see [P] for the word sinnloss. For instance, continued fractions $[0; \gamma_1, \gamma_2, \ldots, \gamma_{3n+2}]$ with $\gamma_i = \sqrt{-1}$ (1≤i≤3n+2, n≥0) are sinnloss, since $\sqrt{-1} + 1/\sqrt{-1} = 0$ and $[0; \sqrt{-1}, \sqrt{-1}, \sqrt{-1}] = 0$. In the case of 1-dimensional continued fractions, for any $\gamma_m \in \mathbb{C}$ (0≤m≤n)

$${}^{TV}[\gamma_0; \gamma_1, \ldots, \gamma_n] = [0; \gamma_n, \gamma_{n-1}, \ldots, \gamma_0] \text{ if } \gamma_0 \ne 0,$$
$${}^{TV}[\gamma_0; \gamma_1, \ldots, \gamma_n] = [0; \gamma_n, \gamma_{n-1}, \ldots, \gamma_2] \text{ if } \gamma_0 = 0$$

hold as far as they are not sinnloss. We denote by $[\underline{\gamma_1}; \underline{\gamma_2}, \ldots, \underline{\gamma_k}]^{(n)}$ the continued fraction

$$[\underbrace{\underline{\gamma_1}; \underline{\gamma_2}, \ldots, \underline{\gamma_k}}_{(1)}, \underbrace{\underline{\gamma_1}, \underline{\gamma_2}, \ldots, \underline{\gamma_k}}_{(2)}, \ldots, \underbrace{\underline{\gamma_1}, \underline{\gamma_2}, \ldots, \underline{\gamma_k}}_{(n)}],$$

which is a periodic continued fraction of length kn-1.

Theorem 15. Let σ_m be $\mathbb{C}$-substitutions defined by

$$\sigma_m(a_i):=a_0^{\gamma_{i,m}}a_{i+1}\ (\ \sigma_m(a_i):=a_{i+1} \text{ if } \gamma_{i,m}=0)\ \text{ for } 0\le i\le s-1, \text{ and } \sigma_m(a_s):=a_0,$$

and let σ be a substitution defined by

$$\sigma:=\sigma_1\sigma_2\cdots\sigma_k.$$

Then

$$1/|\sigma^n(a_0)|_{a_0}\cdot(|\sigma^n(a_0)|_{a_1}, |\sigma^n(a_0)|_{a_2}, \dots, |\sigma^n(a_0)|_{a_s}) = {}^{TV}[\ \underline{\gamma_k}; \underline{\gamma_{k-1}}, \dots, \underline{\gamma_1}]^{(n)},$$

where

$$\underline{\gamma_m}:= {}^t(\gamma_{s-1,m}, \gamma_{s-2,m}, \dots, \gamma_{0,m}).$$

In particular, the direction

$$\underline{\delta}(\sigma; a_0)=\lim_{n\to\infty} 1/|\sigma^n(a_0)|_{a_0}\cdot(|\sigma^n(a_0)|_{a_0}, |\sigma^n(a_0)|_{a_1}, \dots, |\sigma^n(a_0)|_{a_s})\in L(\sigma)^{s+1}$$

exists iff the limit of the transposed value

$${}^{TV}CF=\lim_{n\to\infty} {}^{TV}[\ \underline{\gamma_k}; \underline{\gamma_{k-1}}, \dots, \underline{\gamma_1}]^{(n)}$$

of a periodic continued fraction $CF=\lim_{n\to\infty}[\ \underline{\gamma_k}; \underline{\gamma_{k-1}}, \dots, \underline{\gamma_1}]^{(n)}$ converges.

Proof. It is clear that $\sigma\in \mathrm{Sub}_{a_0^z}(\mathscr{A}^{\mathbb{C}^\times})$ $(\exists z\in\mathbb{C}^\times)$, so that the normalized substitution of the σ has a unique fixed point prefixed by $a=a_0$. In particular, $|\sigma^n(a_0)|_{a_0}\neq 0$ for all $n\ge 0$. In view of Lemma 13, we have ${}^\times M_{\sigma_m}= {}^tC(\underline{\gamma_m})$, so that

$$\begin{aligned}{}^\times M_\sigma &= {}^\times M_{\sigma_1\sigma_2\cdots\sigma_k}= {}^\times(M_{\sigma_1}M_{\sigma_2}\cdots M_{\sigma_k})= {}^\times M_{\sigma_1}\,{}^\times M_{\sigma_2}\cdots{}^\times M_{\sigma_k}\\ &= {}^tC(\underline{\gamma_1})\,{}^tC(\underline{\gamma_2})\cdots{}^tC(\underline{\gamma_k})= {}^t(C(\underline{\gamma_k})C(\underline{\gamma_{k-1}})\cdots C(\underline{\gamma_1}))\end{aligned}$$

Hence we get

$${}^\times M_{\sigma^n}= {}^\times((M_\sigma)^n)=({}^\times M_\sigma)^n=(\ {}^t(C(\underline{\gamma_k})C(\underline{\gamma_{k-1}})\cdots C(\underline{\gamma_1}))\ )^n= {}^t((C(\underline{\gamma_k})C(\underline{\gamma_{k-1}})\cdots C(\underline{\gamma_1}))^n\ ),$$

i.e.,

$$M_{\sigma^n}= {}^\times(\ {}^t((C(\underline{\gamma_k})C(\ \underline{\gamma_{k-1}})\cdots C(\ \underline{\gamma_1}))^n))$$

Therefore, by setting

$$(r_{j,n}^{(i)})_{0\le i\le s, 0\le j\le s}:=(C(\underline{\gamma_k})C(\ \underline{\gamma_{k-1}})\cdots C(\ \underline{\gamma_1}))^n,$$

we get

$$(|\sigma^n(a_0)|_{a_0}, |\sigma^n(a_0)|_{a_1}, \dots, |\sigma^n(a_0)|_{a_s})=(r_{s,n}^{(s)}, r_{s-1,n}^{(s)}, \dots, r_{0,n}^{(s)}),$$

which implies that $\lim_{n\to\infty} 1/|\sigma^n(a_0)|_{a_0}\cdot(|\sigma^n(a_0)|_{a_0}, |\sigma^n(a_0)|_{a_1}, \dots, |\sigma^n(a_0)|_{a_s})$ converges to $\underline{\delta}(\sigma; a_0)$ iff $\lim_{n\to\infty} 1/r_{s,n}^{(s)}\cdot(r_{s,n}^{(s)}, r_{s-1,n}^{(s)}, \dots, r_{0,n}^{(s)})$ converges and

$$\begin{aligned}(\delta_1, \dots, \delta_s)&= \lim_{n\to\infty} 1/|\sigma^n(a_0)|_{a_0}\cdot(|\sigma^n(a_0)|_{a_1}, \dots, |\sigma^n(a_0)|_{a_s})\\ &= \lim_{n\to\infty} {}^{TV}[\ \underline{\gamma_k}; \underline{\gamma_{k-1}}, \dots, \underline{\gamma_1}]^{(n)}. \blacksquare\end{aligned}$$

Such as complex continued fraction algorithms over a quadratic number field, "relative" continued fractions over a number field K, i.e., their partial denominators $\underline{\gamma_n}\in\mathrm{Ord}(K)$ will be of interest related to number theory, cf. [H], [TY1]. Among them, it will be an interesting problem to find algorithms such that the resulting continued expansions of algebraic elements$\in K^s$ (or K^{s+1}) always become periodic and Diophantine, i.e., the characteristic polynomial$\in$ $\mathrm{Ord}(K)[x]$ of the matrix $\mathrm{Per}\in\mathfrak{M}_{s+1}(K)$ coming from the period is irreducible over K and

$$|\lambda^\#|>1>|\lambda_\#|, \text{ or } 1>|\lambda^\#|^2>|\lambda_\#|$$

holds for the dominant and subdominant eigenvalues of Per. We gave a candidate of such an algorithm for $K=\mathbb{Q}$, $s=2$, cf. [FISTY]. The matrices M_σ in Lemma 13 and Theorem 15 are unimodular matrices$\in\mathfrak{M}_{s+1}(\mathbb{Z})$, i.e., $\det M_\sigma=\pm1$. We can extend Lemmas 9, 10 to substitutions having unimodular M_σ over K, i.e., $M_\sigma\in\mathfrak{M}_{s+1}(\mathrm{Ord}(K))$ and $\det M_\sigma\in\mathrm{Ord}^\times(K)$ (:=the unit group of K) by considering relative continued fractions over K, i.e., continued fractions with partial denominators $\underline{\gamma_n}\in\mathrm{Ord}(K)$ and partial "numerators" $\nu_n\in\mathrm{Ord}^\times(K)$. Such a continued fraction may be of particular interest related to Diophantine approximation. It will be also interesting to study intermediate convergents $\underline{\pi}(w(\sigma, a), n)$ for substitutions $\sigma\in\mathbf{Sub}_a(\mathscr{A}(\mathrm{Ord}(K)\backslash\{0\})$, cf. (i-iii), Fig. 8, Example 1 in Section 7. In this paper, we do not go further related to continued fractions.

§7. Experiments

In this section, we give some expamples of $\mathbb{R}$-substitutions over $\{a, b, c\}$^$\mathbb{R}^\times$, and $\mathbb{C}$-substitutions over $\{a, b\}$^$\mathbb{C}^\times$ according to the definitions given in Sections 3, 4. For a dominant substitution σ over $\{a, b, c\}$^$\mathbb{R}^\times$, the Rauzy set $R(w)=R(w(\sigma^*, a))$ lies on the 2-dimensional subspace $\Pi(\underline{\delta}(w))$ over $\mathbb{R}$ of the Euclidean space $\mathbb{R}^3$; and for a dominant substitution σ over $\{a, b\}$^$\mathbb{C}^\times$, the Rauzy set $R(w)=R(w(\sigma^*, a))$ lies on the 1-dimensional subspace $\Pi(\underline{\delta}(w))$ over $\mathbb{C}$ of the unitary space $\mathbb{C}^2$, where σ^* is the normalized substtitution of σ. The Rauzy sets lie on 2-dimensional spaces over $\mathbb{R}$ for both cases. We will have a look for some Rauzy sets by experiments. To be acurate, they are not Rauzy sets, but

$$R((\sigma^*)^n(a)):=\mathrm{proj}_{\underline{\delta}(w)}PR((\sigma^*)^n(a))$$

for a number n such that $PR((\sigma^*)^n(a))$ consists of sufficiently many points. It is unclear when $R(w)$ is a bounded set. We also consider the case where $R(\lim (\sigma^*)^n(a))$ may not bounded for some dominant substitutions.

§7.1. $\mathbb{R}$-substitutions

In this subsection, we give some experiments on substitutions over $\{a, b, c\}$^$\mathbb{R}_{>0}$ having the fixed point $w=\lim \sigma^n(a)$ with the direction $\underline{\delta}(w)\in\mathbb{R}_{>0}{}^3$. In all the figures on $\Pi(\underline{\delta}(w))\subset\mathbb{R}^3$, the directions of axes are canonical, i.e., the direction of horizontal axis coincides with a vector $\underline{\beta}=(\beta_0, \beta_1, 0)$ $(\beta_0<0)$ orthogonal to the direction $\underline{\delta}(w)$, and the vertical axis coincides with a vector $(\gamma_0, \gamma_1, \gamma_2)$ $(\gamma_2>0)$ which is orthogonal to $\underline{\delta}(w)$ and $\underline{\beta}$, cf. Section 5 for notation.

Example 1. Let σ_t be a substitution over $\{a, b, c\}$^$\mathbb{R}_{>0}$ defined by

$$\sigma_t(a):=ab,\ \sigma_t(b):=a^t c\ (0<t\leq 1,\ \sigma_0(b):=c),\ \sigma_t(c):=a\ (a=a_0, b=a_1, c=a_2).$$

Recall the definitions: $R(w)$, $R(w; a_i, *)$, $R(w; *, x)$, and $R(w; a_i, x)$; $R(w; a_i, *)$ (resp., $R(w; *, x)$, and $R(w; a_i, x)$) is the $(a_i, *)$-part (resp., $(*, x)$-part, (a_i, x)-part) of the Rauzy set of $w=\lim \sigma^n(a)$, cf. Section 5. One can modify the definitions of $R(w)$, $R(w; a_i, *)$, $R(w; *, x)$, and $R(w; a_i, x)$ for a finite word w. For instance,

$$R(\sigma_t{}^n(a); *, x) := \mathrm{proj}_{\underline{\delta}(w)}\{\underline{p}(\sigma_t{}^n(a), m);\ 1\leq m\leq|\sigma_t{}^n(a)|_{\mathrm{symb}}$$
$$\&\ \text{the length of the segment } (\underline{p}(\sigma_t{}^n(a), m-1), \underline{p}(\sigma_t{}^n(a), m)]=x\}.$$

Taking n=20, we can watch the eight animations each of which consits of 257 pictures:

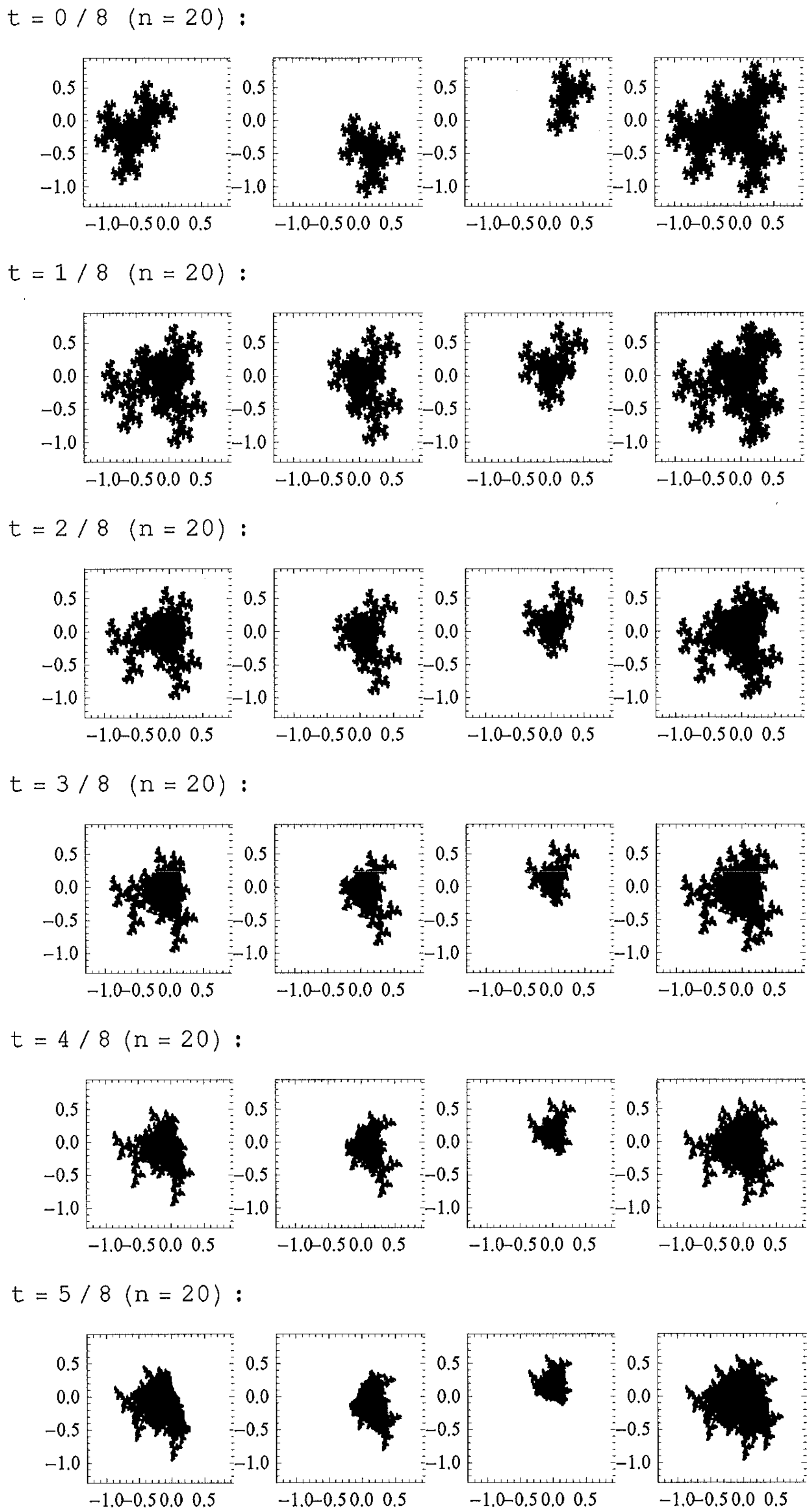
$$R(\sigma_t{}^n(a)),\quad R(\sigma_t{}^n(a); a_i, *)\ (i=0, 1, 2),\ R(\sigma_t{}^n(a); *, 1),\ R(\sigma_t{}^n(a); a_i, 1)\ (i=0, 1, 2)$$

for t=k/256, k=0, 1,..., 256. By such experiments, it seems very likely that the eight maps

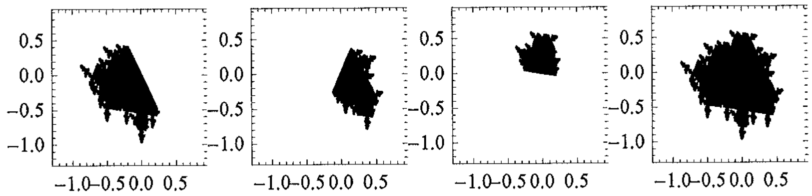
$$[0,1]\ni t\mapsto R(w(t))\subset\mathbb{R}^2,\quad (0,1]\ni t\mapsto R(w(t); a_i, *)\subset\mathbb{R}^2\ (i=0,1,2),$$
$$[0,1)\ni t\mapsto R(w(t); *, 1)\subset\mathbb{R}^2,\quad [0,1)\ni t\mapsto R(w(t); a_i, 1)\subset\mathbb{R}^2\ (i=0, 1, 2),\ w(t)=\lim \sigma_t{}^n(a)$$

are "continuous"; and we can guess that the maps for $w(t)=\lim \sigma_t{}^n(a)$ are also continuous. Concerning such phenomena, we give a problem which asks a suitable metric or topology for the sets of finite/infinite points. It is interesting that points$\in R(\sigma_t{}^{20}(a); *, x)$ $(0<x<1)$ move faster than points$\in R(\sigma_t{}^{20}(a); *, 1)$ when t runs from 1/k to (k-1)/k. Fig. 1 shows four slides (i.e., a part of the 4 animations of $R(\sigma_t{}^{20}(a); a_i, *)$ (i=0, 1, 2) and $R(\sigma_t{}^{20}(a))$ (from the left to the right; the order is the same for Fig. 2, Fig. 3) with x=k/8, k=0, 1,..., 8. Concerning Fig. 1, we remark that $R(\sigma_0{}^{20}(a))$ consists of $|\sigma_0{}^{20}(a)|_{\mathrm{symb}}=2745$ points, and $R(\sigma_t{}^{20}(a))$ consists of $|\sigma_t{}^{20}(a)|_{\mathrm{symb}}=223317$ points for each t=k/8 $(1\leq k\leq 8)$. Thus, the set $R(\sigma_0{}^{20}(a))$ looks sparser than $R(\sigma_1{}^{20}(a))$. If we ignore the density of points and the details of the contours, the shapes $R(\sigma_0{}^{20}(a))$ and $R(\sigma_0{}^{31}(a))$ (see Fig. 2) are not able to be distinguished by their look as well as the shapes $R(\sigma_1{}^{20}(a))$ and the well-known "Arnoux-Rauzy fractal $R(\lim \sigma_1{}^n(a))$ are. One can also observe the discontinuity; it seems very likely that $R(\sigma_t{}^n(\mathbf{a}); \mathbf{a}_i, *)$ is discontinuos at t=0 from the right; Fig. 3 shows the jumps between $R(\sigma_0{}^{20}(\mathbf{a}); \mathbf{a}_i, *)$ and $R(\sigma_{1/1024}{}^{20}(\mathbf{a}); \mathbf{a}_i, *)$, i=0, 1, 2; while the $(\mathbf{a}_i, 1)$-part has not such a jump for all i=0, 1, 2, cf. Fig. 4.

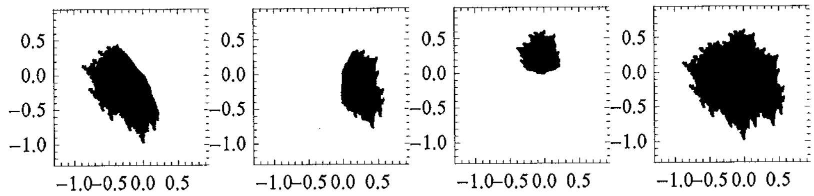
Fig. 1.



t = 6 / 8 (n = 20) :



t = 7 / 8 (n = 20) :



t = 8 / 8 (n = 20) :

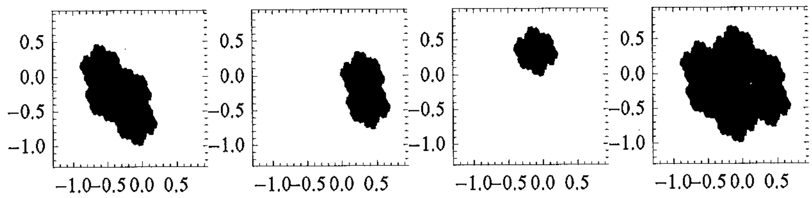


Fig. 2. t = 0 (n = 31) :

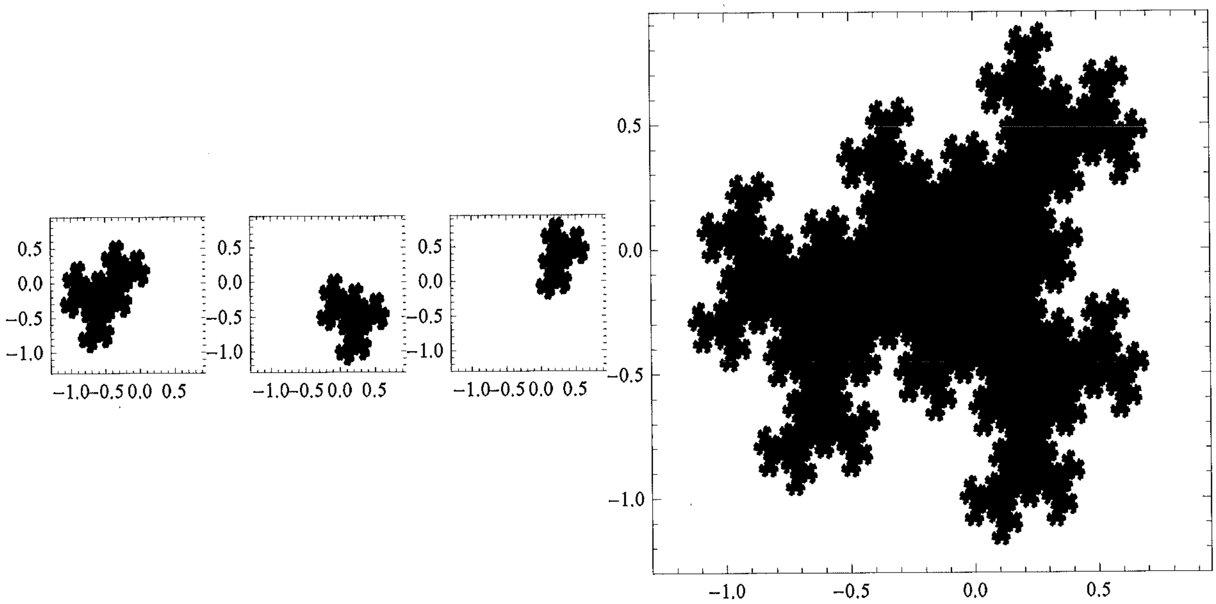
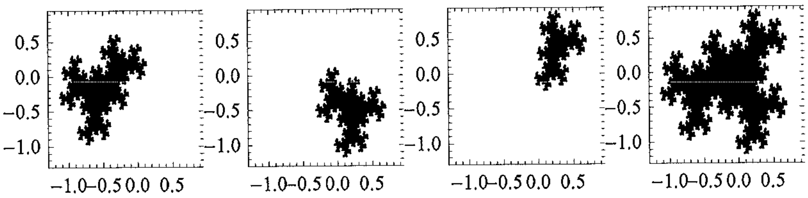


Fig. 3. R (σ_t^n (a) ; a_i, *), i = 0, 1, 2, and R (σ_t^n (a))
t = 0 (n = 20) :



t = 1 / 1024 (n = 20) :

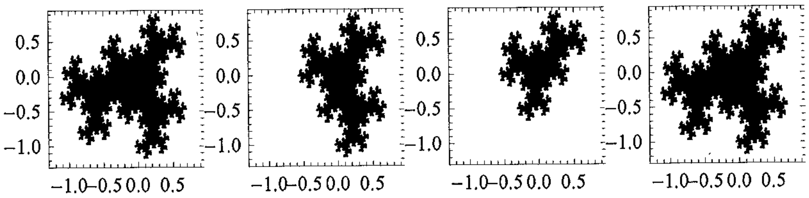
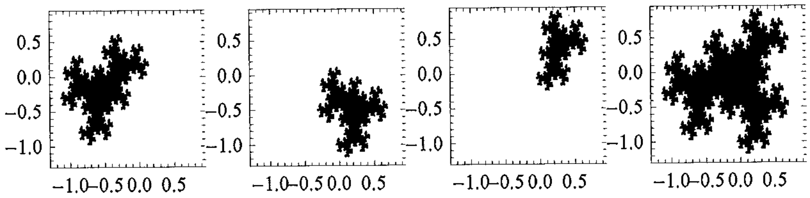
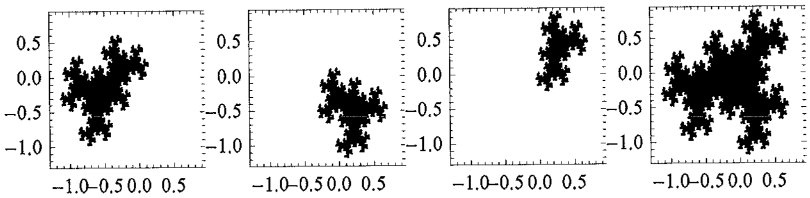


Fig. 4. R (σ_t^n (a) ; a_i, 1), i = 0, 1, 2, and R (σ_t^n (a) ; *, 1)
t = 0 (n = 20) :



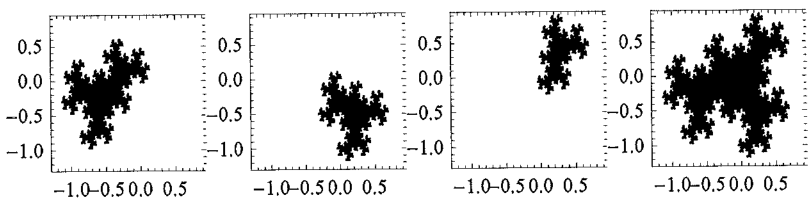
t = 1 / 1024 (n = 20) :



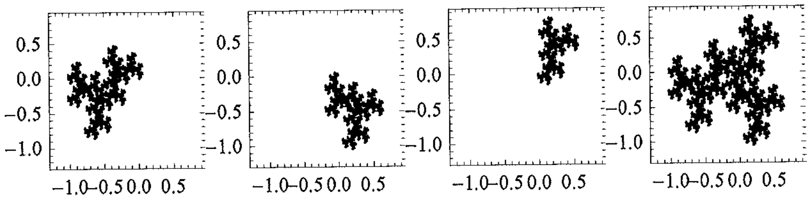
The following Fig. 5 shows 4 slides of R(σ_t^{20}(a); a_i, 1) (i=0,1, 2) and R(σ_t^{20}(a); *, 1) (from the left to the right) with t=k/8, k=0, 1,..., 8. (#R(σ_t^{20}(a); a, 1), #R(σ_t^{20}(a); b, 1), #R(σ_t^{20}(a); c, 1))=(1278, 872, 595) and #R(σ_t^{20}(a); *, 1)=2745 (∀t=k/8, k=0, 1,..., 7); while (#R(σ_1^{20}(a); a, 1), #R(σ_1^{20}(a); b, 1), #R(σ_1^{20}(a); c, 1))=(121415, 66012, 35890), and R(σ_1^{20}(a); *, 1)=223317, where # indicates the number of elements of a finite set.

Fig .5.

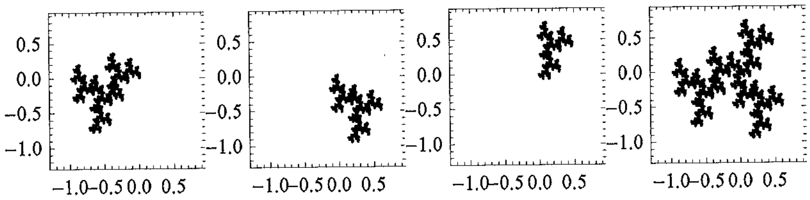
t = 0 / 8 (n = 20) :



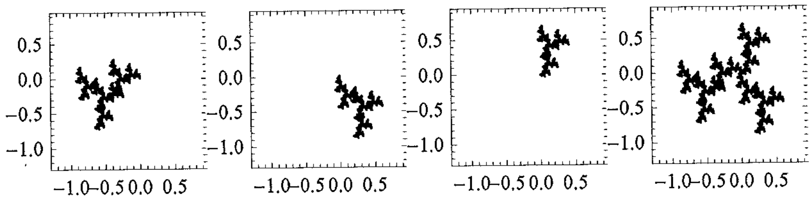
t = 1 / 8 (n = 20) :



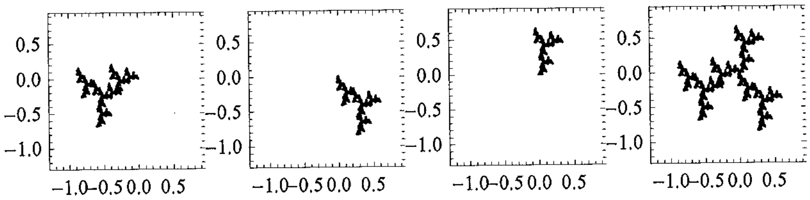
t = 2 / 8 (n = 20) :



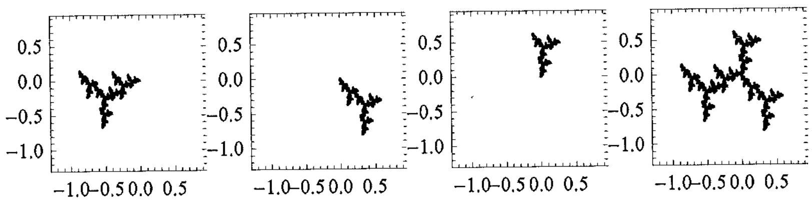
t = 3 / 8 (n = 20) :



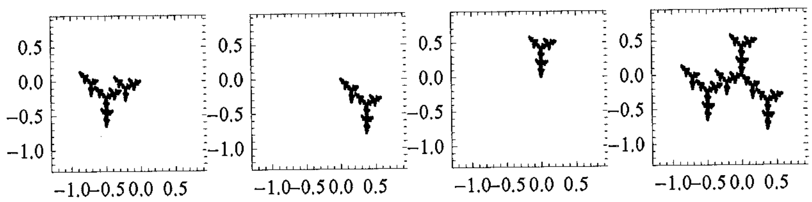
t = 4 / 8 (n = 20) :



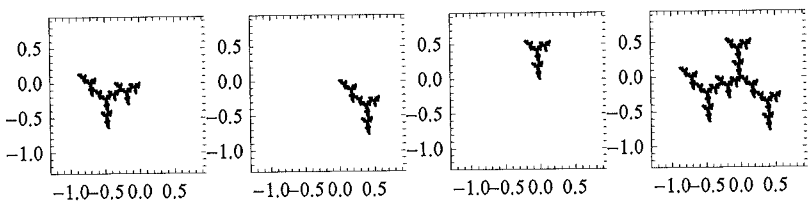
t = 5 / 8 (n = 20) :



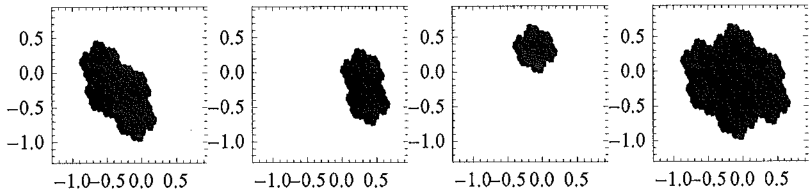
t = 6 / 8 (n = 20) :



t = 7 / 8 (n = 20) :

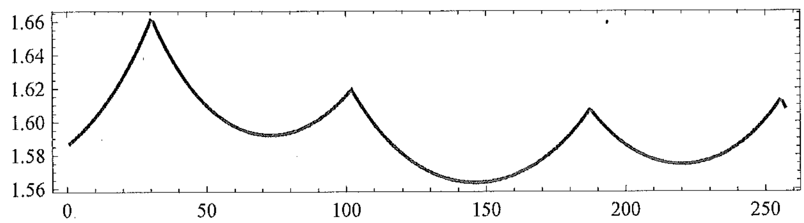


t = 8 / 8 (n = 20) :



From the look of the shapes in Fig. 5, one can guess that the three sets, R $(\sigma_t;\ a_i,\ 1)$ (i=0,1,2) are disjoint except for their boundary for all $0 \le t \le 1$. If so, it will be an interesting phenomenon in compare with that the three sets R $(\sigma_t^{20}(a_0);\ a_i,\ *)$ (i=0, 1, 2) are very likely to be "overlapped" for $0<t<1$ by their look. Recall the definition of $\nu^{(i)}(n)=\mu^{(i)}(|\sigma^n(a_0)|_{symb})$ $(1 \le i \le s,\ n \ge n_0(\sigma))$ with $\mu^{(i)}(n)=\ \mu^{(i)}(w(\sigma,\ a_0);\ n)$ given by (8). We set $\nu_t^{(i)}(n)=\mu^{(i)}(|\sigma_t^n(a_0)|_{symb})$ $(i=1,2,\ 0 \le t \le 1)$. Fig. 6 is the graph of $\underset{i=1,2}{\mathrm{Min}}\ \nu_{T/256}^{(i)}(16)$ for T=0, 1, ..., 256.

Fig. 6 :

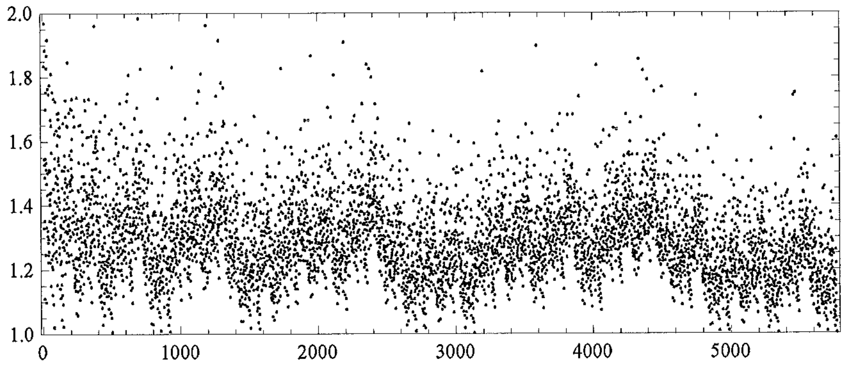


We consider the special case of Example 1. Let σ be a substitution over $\{a, b, c\}^\wedge\mathbb{R}^\times$ defined by

$$\sigma(a):=ab,\quad \sigma(b):=a^{\left(\sqrt{5}-1\right)/2}c,\quad \sigma(c):=a,$$

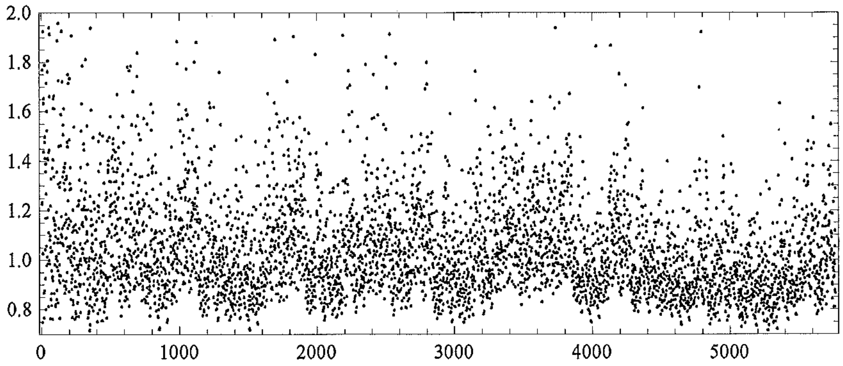
and $w=\lim \sigma^n(a)$ be its fixed point. Then $F(\sigma)=K(\sigma)=\mathbb{Q}(\sqrt{5})$, and $\varphi_{M_\sigma}\in F[x]$ is #-Diophantine over F. The denominator and numerator of the intermediate convergent $\underline{\pi}(w,\ n)$ are algebraic integers$\in \mathrm{Ord}(\mathbb{Q}(\sqrt{5}))$ for all n. We can compare the graphs of the functions $\mu^{(i)}(n)$, $\underset{i=1,2}{\mathrm{Min}}\ \mu^{(i)}(n)$; or $\mu_H^{(i)}(n)$, $\underset{i=1,2}{\mathrm{Min}}\ \mu_H^{(i)}(n)$. We give some part of them: Fig. 7 is the graph of $\underset{i=1,2}{\mathrm{Min}}\ \mu^{(i)}(n)$ for $5 \le n \le 5768$ (the values>2 are excluded. $\underset{i=1,2}{\mathrm{Min}}\ \mu^{(i)}(n)$ sporadically takes values>2). Fig. 8 is the graph (i) $\mu_H^{(1)}(n)$, (ii) $\mu_H^{(2)}(n)$, (iii) $\underset{i=1,2}{\mathrm{Min}}\ \mu_H^{(i)}(n)$ for $5 \le n \le 5768$ (the values$\notin[7/10,2]$ are excluded). It is interesting in connection with (simultaneous) diopantine approximations that they sporadically take bigger values>3/2.

Fig. 7.



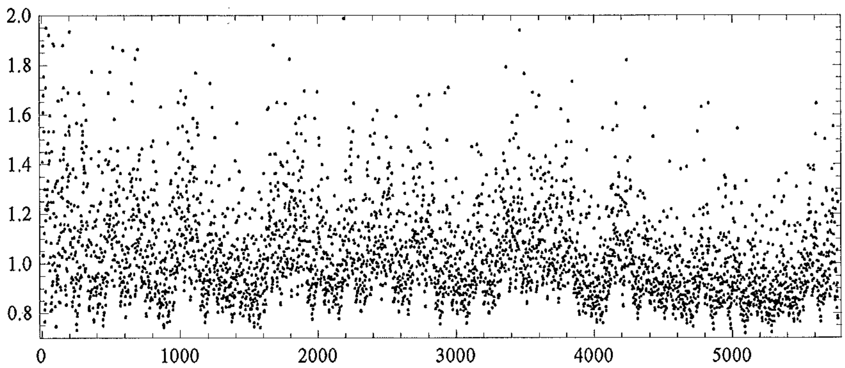
(Max, Min) = (2.72963, 1.00131)

Fig. 8. (i)



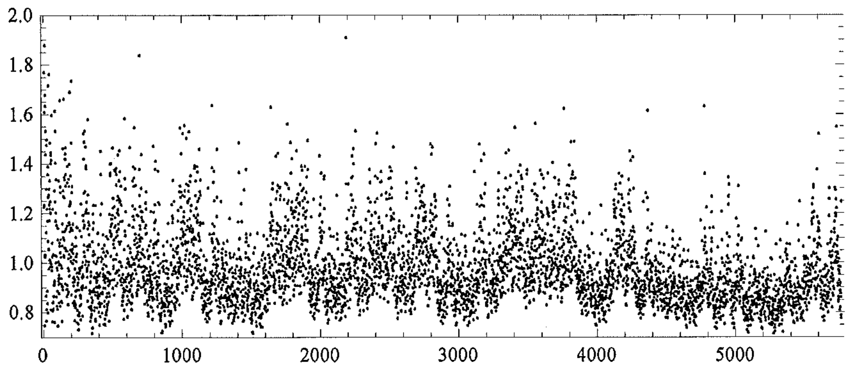
(Max, Min) = (2.3864, 0.686679)

Fig. 8. (ii)



(Max, Min) = (2.70229, 0.719161)

Fig. 8 (iii)



(Max, Min) = (2.04979, 0.686679)

§7.2. ℂ-substitutions

In this subsection, we give some experiments on ℂ-substitutions over $\{a, b\}^{\wedge}\mathbb{C}^{\times}$; the set $\Pi(\underline{\boldsymbol{\delta}}(w))(\subset\mathbb{C}^2)$ can be identified with ℂ. So that all the figures on $\Pi(\underline{\boldsymbol{\delta}}(w))$ can be considered to be subsets of ℂ; the directions of axes are canonical, i.e., the coordinate of horizontal (resp., vertical) axis indicates the real (resp., imaginary) part of complex numbers as usual. In this subsection, we put $\mu(\mathrm{n}):=\mu^{(1)}(\mathrm{n})$.

Example 2. Let σ be a substitution defined by $\sigma(\mathtt{a})=\mathtt{ab}$, $\sigma(\mathtt{b})=\mathtt{a}^{(1+\sqrt{-3})/2}\mathtt{a}^{(1-\sqrt{-3})/2}$. In this case $M_\sigma=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is a real matrix, which is the same as the incidence matrix of Fibonacci substitution. The fixed point w=lim $\sigma^{\mathrm{n}}(\mathtt{a})$ is a spacially unbounded word (i.e., PR(w)$\subset\mathbb{C}^2$ is unbounded) with $\underline{\delta}(\mathrm{w})\in\mathbb{R}^2$. Fig. 9 shows the projection($\subset\mathbb{C}$) of the set PR($\sigma^{18}(\mathtt{a})$)$\subset\mathbb{C}^2$ along a vector (0,1). Fig. 10 is the sets (i) A:=R($\sigma^{18}(\mathtt{a})$; $\mathtt{a}$, *), (ii) B:=R($\sigma^{18}(\mathtt{a})$; $\mathtt{b}$, *), (iii) R($\sigma^{18}(\mathtt{a})$). It is interesting that an "eddy" appears in R($\sigma^{18}(\mathtt{a})$)=A$\cup$B, while we can not see an eddy neither in A, nor B. Fig. 10 (iv) shows a detail of R($\sigma^{18}(\mathtt{a})$). We checked that $\mu(\mathrm{n})>1.9776...$ for all $1\leq \mathrm{n}\leq |\sigma^{18}(\mathtt{a})|_{\mathtt{symb}}=262144$, cf. Fig. 13, 15, 17, 20, 28.

Fig. 9 :

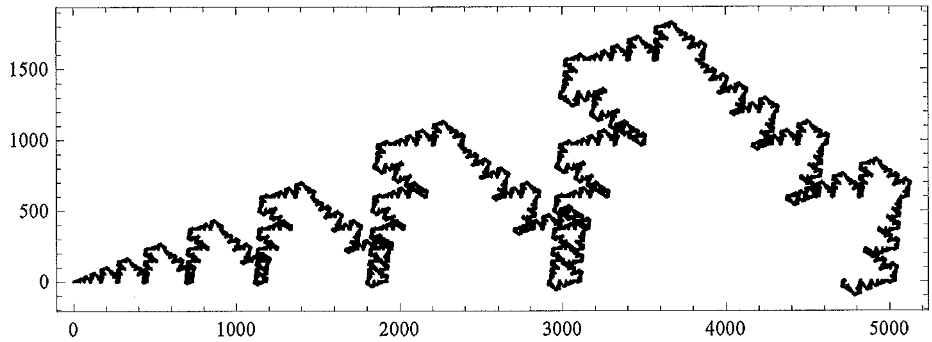


Fig. 10. (i) :

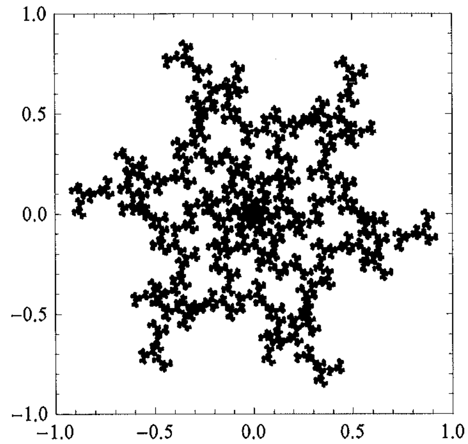


Fig. 10. (ii) :

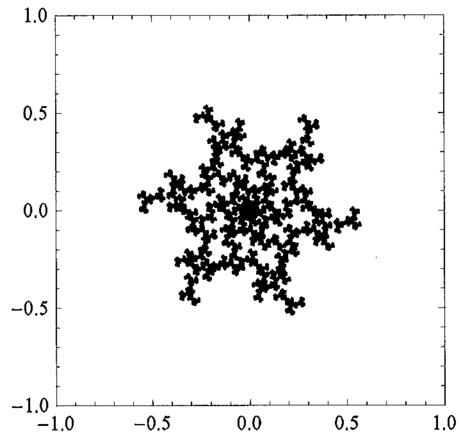


Fig. 10. (iii) :

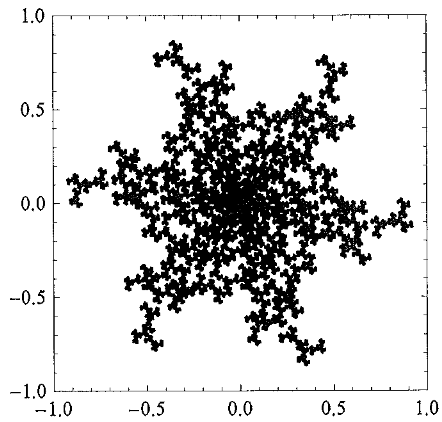
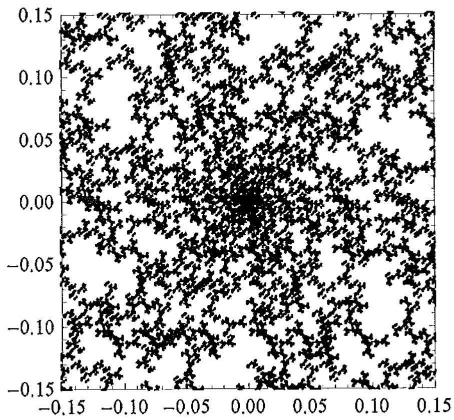


Fig .10. (iv) :



Example 3. Let σ be a substitution defined by $\sigma(\mathtt{a})=\mathtt{a}^{3/5-4\sqrt{-1}/5}\mathtt{b}\mathtt{a}^{3/5+4\sqrt{-1}/5}$, $\sigma(\mathtt{b})=\mathtt{a}$. The σ has no fixed point, and $M_\sigma=\begin{pmatrix} 6/5 & 1 \\ 1 & 0 \end{pmatrix}$ is a real matrix. The normalized substitution σ^* of σ is given by

$$\sigma^*(\mathtt{a})=\mathtt{a}\mathtt{b}^{3/5+4\sqrt{-1}/5}\mathtt{a}^{-7/25+24\sqrt{-1}/25},\ \sigma^*(\mathtt{b})=\mathtt{a}^{3/5+4\sqrt{-1}/5},$$

$$M_{\sigma^*}=\begin{pmatrix} 18/25+24\sqrt{-1}/25 & 3/5+4\sqrt{-1}/5 \\ 3/5+4\sqrt{-1}/5 & 0 \end{pmatrix}.$$

and the fixed point w of σ^* is an unbounded word. Fig.11 (i) (resp., (ii)) shows the projection of the set $PR(\sigma^{*15}(\mathtt{a}))(\subset\mathbb{C}^2)$ along a vector (0,1) (resp., (1,0)). $\underline{\underline{\delta}}(\mathrm{w})\in\mathbb{R}^2$ holds as well as Example 2. Fig. 12 shows the set $R(\sigma^{*15}(\mathtt{a}))$.

Fig. 11 (i) (ii)

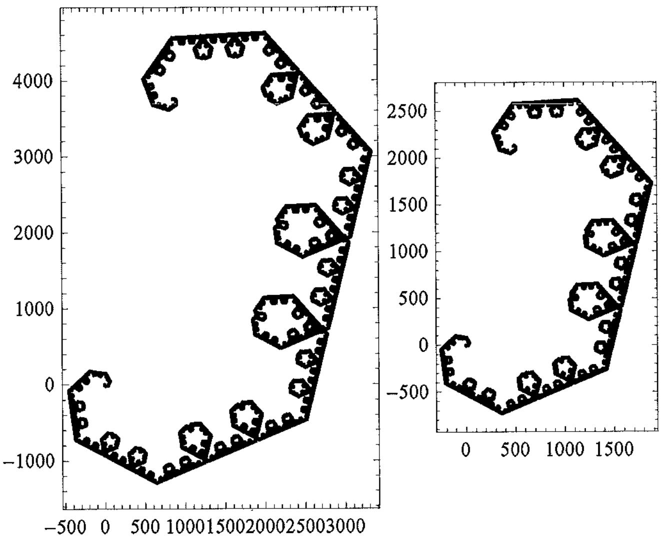
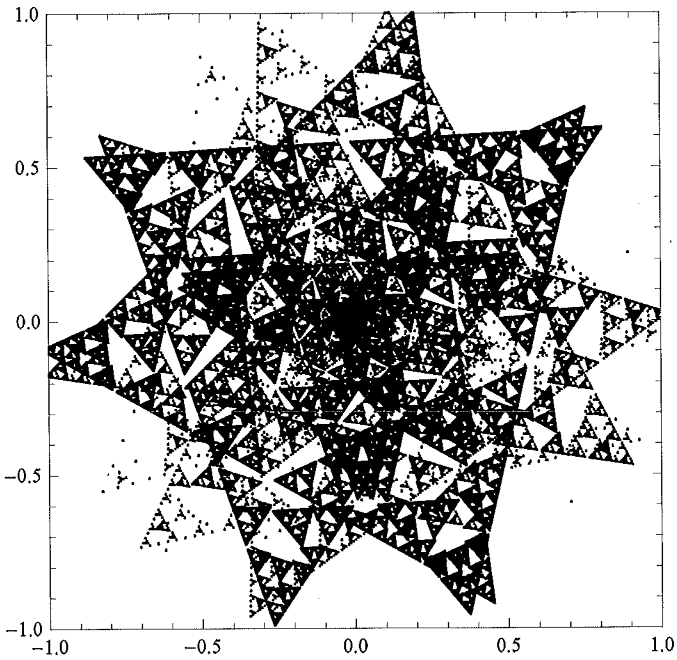
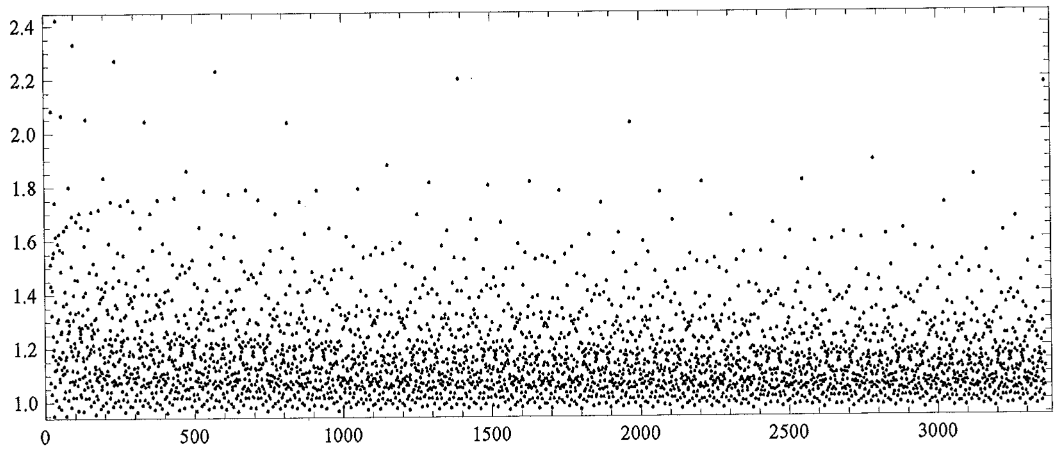


Fig. 12:



The shapes (i), (ii), Fig. 11 can not be similar since they have distinct number of points, but they look similar. Such a phenomenon comes from the fact that the intermediate convergents approximate the direction $\underline{\underline{\delta}}$(w) well, cf. Fig. 27 (i), (ii) in Example 9. Fig.13 is the graph of μ(n) (0.9507...<μ(n)<2.4216... for $19 \leq n \leq |\sigma^{*9}(a)|_{symb}=3363$). We can guess that $\liminf_{n\to\infty} \mu(n) = 1$.

Fig. 13 :



Example 4. Let σ be a substitution defined by $\sigma(\mathtt{a})=\mathtt{aba}$, $\sigma(\mathtt{b})=\mathtt{b}^{3/5}\mathtt{a}^{(1-\sqrt{-1})/\sqrt{2}}\mathtt{b}^{3/5}$. $M_\sigma = \begin{pmatrix} 2 & (1-\sqrt{-1})/\sqrt{2} \\ 1 & 6/5 \end{pmatrix}$, and the fixed point w of σ is a spacially unbounded word with $\underline{\underline{\delta}}(w)\notin\mathbb{R}^2$. Fig. 14 shows the set (i) $R(\sigma^{18}(\mathtt{a}))$ and (ii) its (*, 1)-part of $R(\sigma^{18}(\mathtt{a}))$. Fig. 15 is the graph of $\mu(n)$ ($1.02258<\mu(n)<2.34245$ for $20\le n\le|\sigma^{10}(\mathtt{a})|_{\mathtt{symb}}=59049$).

We can guess that $\liminf_{n\to\infty}\mu(n)=1$ as well as Example 3. The difference from the previous example may be that $\mu(n)>1$ for all $n\ge n_0(\sigma)$ possibly holds. Thus, this is possibly the case where the convergence is strong including intermediate convergents.

Fig. 14.

(i) (ii)

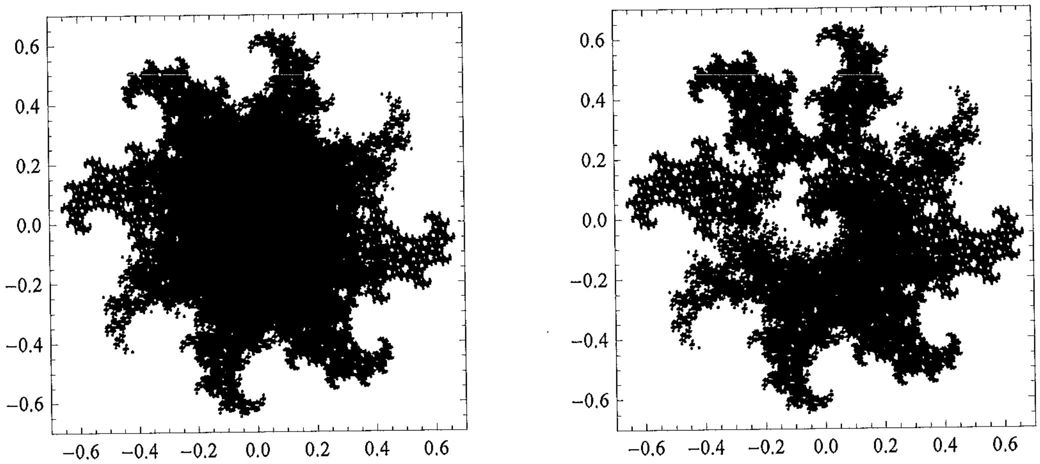
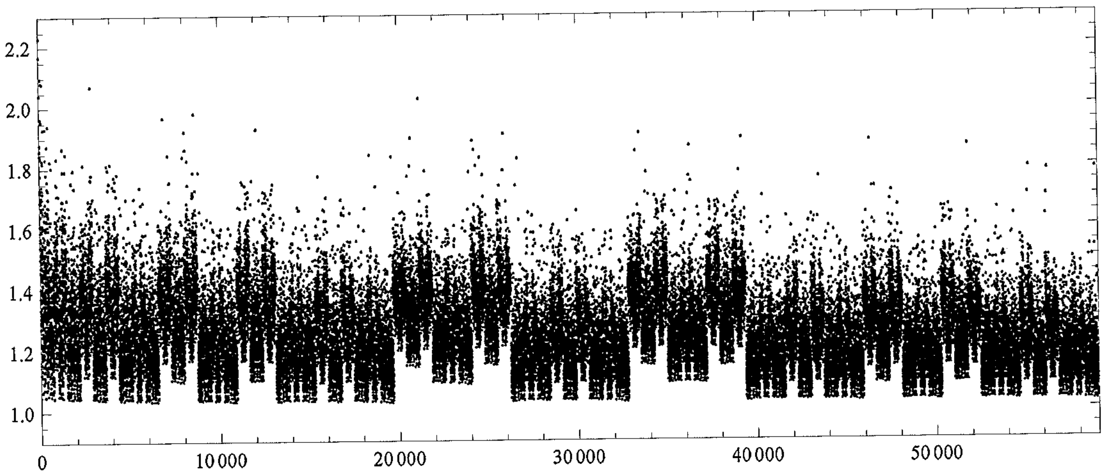


Fig. 15 :



Example 5. Let σ be a substitution defined by $\sigma(a)=a\,a^{e^{-3\pi\sqrt{-1}/8}}\,b^{e^{-\pi\sqrt{-1}/8}}$, $\sigma(b)=a^{e^{-3\pi\sqrt{-1}/8}}$.

$M_\sigma=\begin{pmatrix} 1+e^{-3\pi\sqrt{-1}/8} & e^{-3\pi\sqrt{-1}/8} \\ e^{-\pi\sqrt{-1}/8} & 0 \end{pmatrix}$. The fixed point w of σ is a spacially unbounded word with imaginary direction $\underline{\underline{\delta}}(w)\notin\mathbb{R}^2$. Fig. 16 shows the set (i) $R(\sigma^{16}(a))$ and its details (ii), (iii). Fig. 17 is the graph of $1.00002<\mu(n)<2.33772$ $(20\le n\le|\sigma^{10}(a)|_{symb}=8119)$.

Fig. 16.

(i) (ii) (iii)

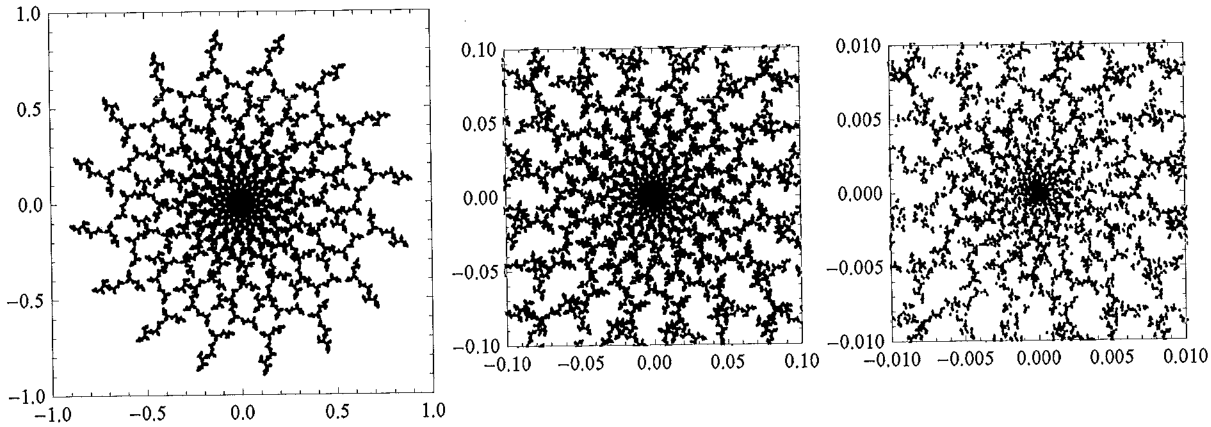


Fig. 17 :

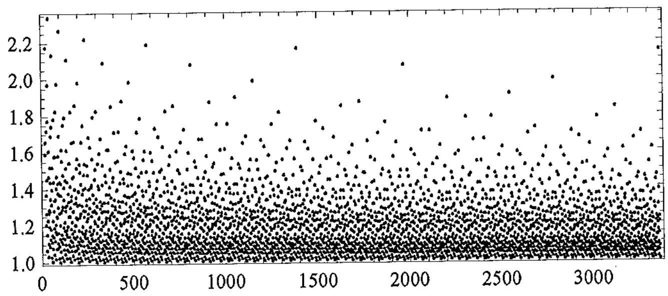
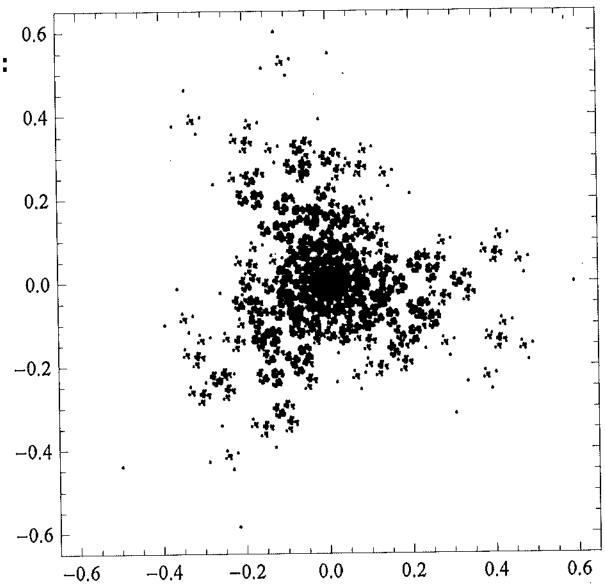


Fig. 18 :



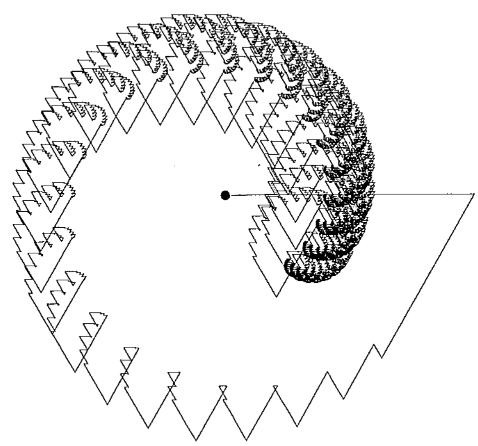
Example 6. Let $\sigma=\sigma_r$ be a substitution defined by $\sigma(a)=ab^{r\left(\sqrt{-3}-1\right)/2}$, $\sigma(b)=a^{r\left(\sqrt{-3}-1\right)/2}$ $(r>0)$. We consider two cases: (1) r=4/5, and (2) r=1/2. Both are Diophantine substitutions with $\underline{\underline{\delta}}(w)\notin\mathbb{R}^2$.

(1) (r=4/5) This is a #-Diophantine case. The fixed point w=lim σ^n(a) is spacially unbounded. Fig. 18. shows the set R(σ^{28}(a)).

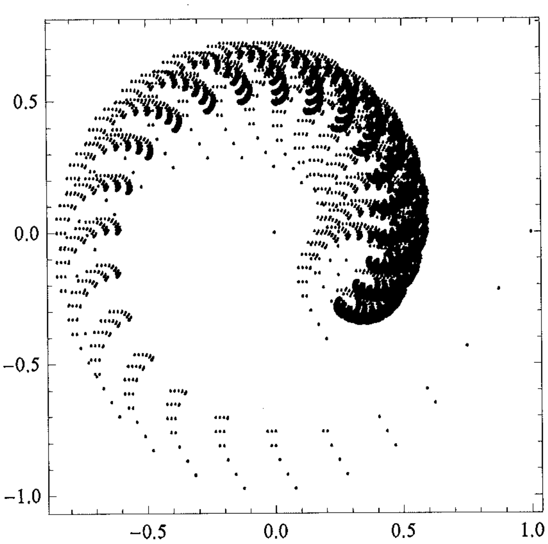
(2) (r=1/2) This is a ♭-Diophantine case. $|\lambda^{\#}|<1$ and the fixed point w=lim σ^n(a) is metrically unbounded, since $a^{\left(r\left(\sqrt{-3}-1\right)/2\right)^2}$ appears infinitely often as a subword of w. The quantity $|w(1)w(2)\cdots w(n)|_{metr}$ tends to infinity slowly. Fig. 19. (i) is the projection of LR(σ^{28}(a)) along a vector (0,1); (ii) (resp., (iii)) is the projection of PR(σ^{28}(a)) along a vector (0,1) (resp., (1,0)) together with the origin. In view of Fig. 19, we conjecture that the fixed point w is a spacially bounded word. Fig. 20 is the graph of $0.335925<\mu(n)<188.860$ for $n_0(\sigma)=3\le n\le|\sigma^{28}(a)|_{symb}=832040$ (the values larger than 50 are excluded). Fig. 21. (resp., Fig. 22) shows the graph of $\log|w(n)|_{metr}$ (resp., $|w(1)w(2)\cdots w(n)|_{metr}$) for $1\le n\le|\sigma^{28}(a)|_{symb}$; they synchronize well with Fig. 20, and Fig. 19 (i). In view of (ii) and (iii) in Fig. 19, we can also see that $\underline{\underline{\delta}}(w)\notin\mathbb{R}^2$, cf. Fig. 11, Eg. 3.

Fig. 19.

(i)



(ii)



(iii)

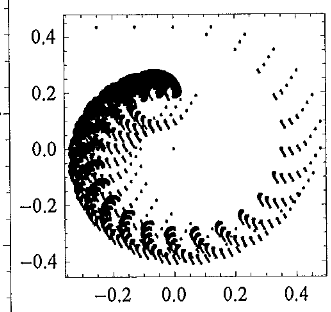


Fig. 20.

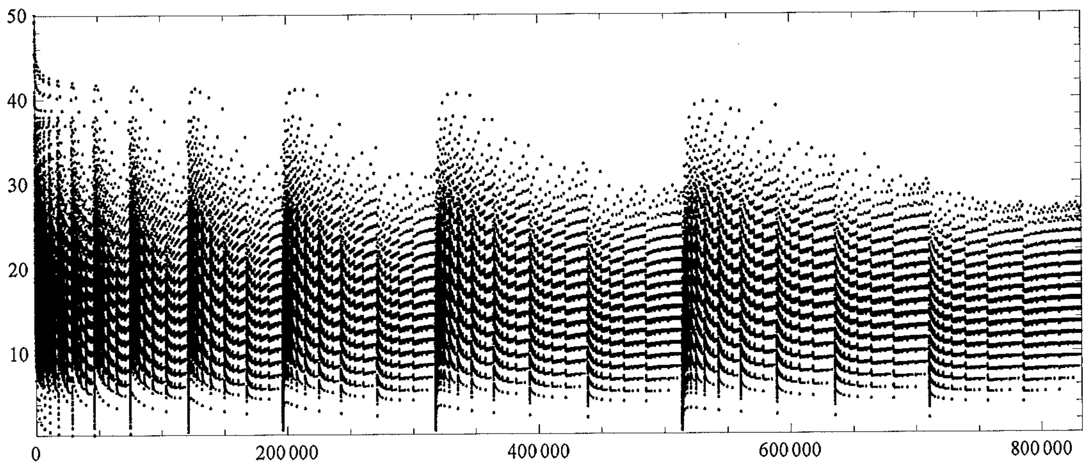


Fig. 21.

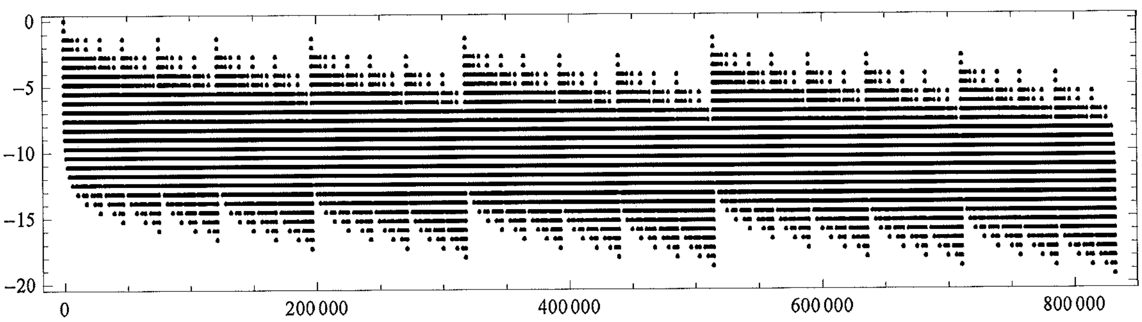
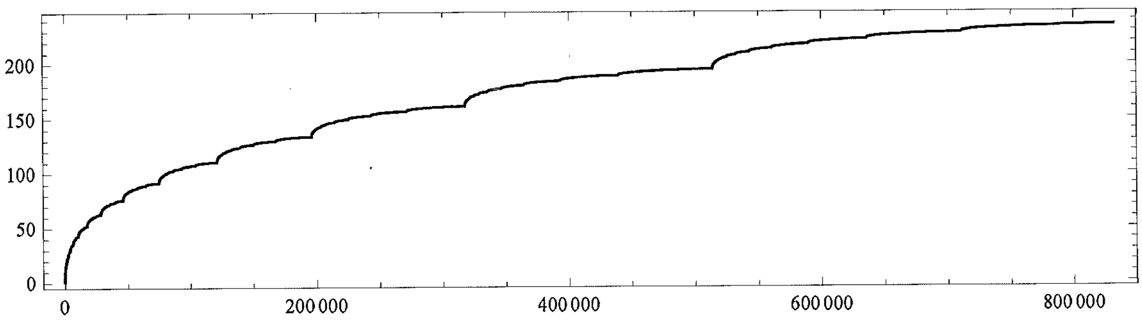


Fig. 22.



Example 7. Let σ be a substitution defined by $\sigma(\mathtt{a})=\mathtt{a}\mathtt{a}^{e^{-5\pi\sqrt{-1}/8}}\mathtt{b}^{e^{-5\pi\sqrt{-1}/8}}$, $\sigma(\mathtt{b})=\mathtt{a}^{e^{-5\pi\sqrt{-1}/8}}$.

$M_\sigma=\begin{pmatrix} 1+e^{-5\pi\sqrt{-1}/8} & e^{-5\pi\sqrt{-1}/8} \\ e^{-5\pi\sqrt{-1}/8} & 0 \end{pmatrix}$ is #-Diophantine over K=K(σ)= $\mathbb{Q}(e^{5\pi\sqrt{-1}/8})$, namely, the characteristic polynomial $\varphi_{M_\sigma}\in$F[x]=E[x]=K[x] is irreducible over E=E(σ), and $|\lambda^{\#}|>1>|\lambda_{\#}|$ holds for the dominant and subdominant eigenvalues. They are algebraic units of degree 16 over $\mathbb{Q}$, since det $M_\sigma=(1-\sqrt{-1})/\sqrt{2}$ is a unit$\in$Ord(K) $^\times$, i.e., M_σ is "unimodular" over K. The fixed point w of σ is spacially unbounded with $\underline{\underline{\delta}}(w)\notin\mathbb{R}^2$. Fig. 23 is the set proj $_{(0,1)}$(L-R(σ^{14}(a)) together with the origin indicated by a bigger dot. Fig. 24 is R(σ^{15}(a)). We guess that the two parts R(w; a, *), R(w; b, *) are overlapped, cf. Example 6.

Fig. 23 :

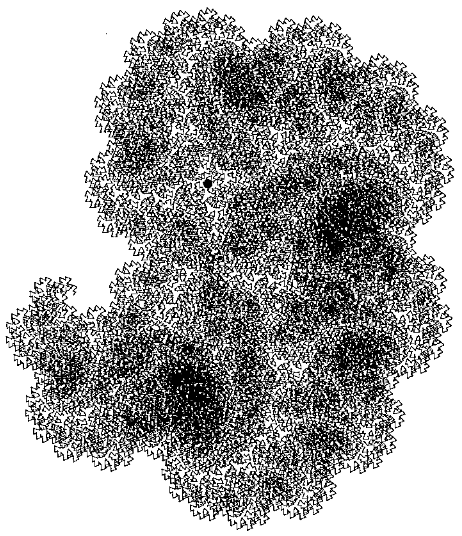
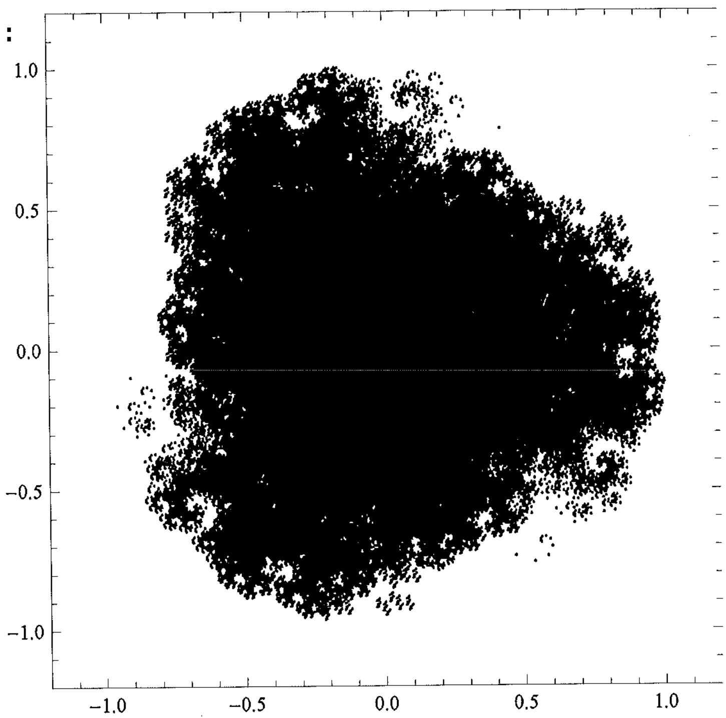


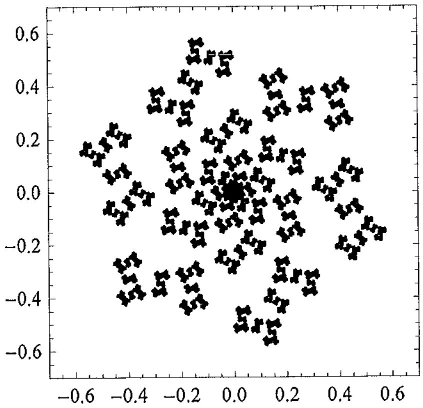
Fig. 24 :



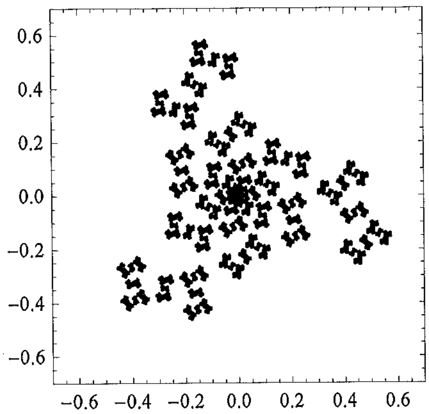
Example 8. Let σ be a substitution defined by $\sigma(\mathrm{a})=\mathrm{aba}$, $\sigma(\mathrm{b})=\mathrm{a}^{e^{2\pi\sqrt{-1}/3}}$. This σ is also #-Diophantine over $K=K(\sigma)=\mathbb{Q}(\sqrt{-3})$, and $\underline{\underline{\delta}}(w)\notin\mathbb{R}^2$ as Example 7. Fig. 25. (i) is the set $R(\sigma^{14}(\mathrm{a}))$, (ii) is the (a,*)-part of $R(\sigma^{14}(\mathrm{a}))$, (iii) is the (b, *)-part of $R(\sigma^{14}(\mathrm{a}))$. In view of them, we can guess that $R(w)=R(w;\ \mathrm{a},\ *)\cup R(w;\ \mathrm{b},\ *)$ is a disjoint union, cf. Example 7. In this case, $1.04360...<\mu(n)<3.39749...$ $(3\leq\forall n\leq|\sigma^{14}(\mathrm{a})|_{\mathrm{symb}}=275807)$. We guess that $\mu(n)>1$ $(\forall n\geq 1)$ and $\liminf_{n\to\infty}\mu(n)=1$.

Fig. 25.

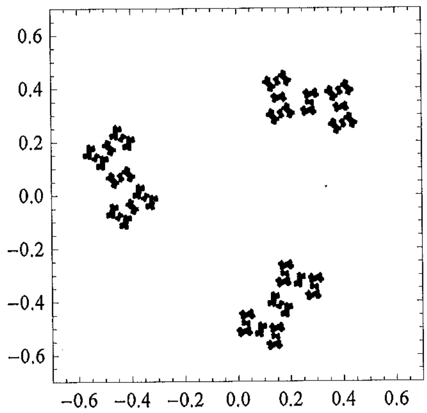
(i)



(ii)



(iii)



Example 9. Let σ be a substitution defined by $\sigma(\mathtt{a})=\mathtt{a}\mathtt{b}^{-1/2-\sqrt{-3}/2}\mathtt{a}^{-1/2+\sqrt{-3}/2}$, $\sigma(\mathtt{b})=\mathtt{b}^{1/2-\sqrt{3}/6-\sqrt{-1}/2-\sqrt{-3}/6}\mathtt{a}^{-1/2-\sqrt{-3}/2}\mathtt{b}^{-1/2-\sqrt{3}/6-\sqrt{-1}/2+\sqrt{-3}/6}$. $K=K(\sigma)=\mathbb{Q}(\sqrt{3},\sqrt{-1})$ is a number field of degree 4 over $\mathbb{Q}$. On the other hand the splitting field $L(\sigma)$ of φ_{M_σ} is of degree 8 over $\mathbb{Q}$, so that φ_{M_σ} is irreducible over K. For eigenvalues of φ_{M_σ}, $|\lambda^{\#}|>|\lambda_{\#}|>1$ holds. Hence σ is not Diophantine, but a dominant substitution so that the direction $\underline{\underline{\delta}}(w)\notin\mathbb{R}^2$ of $w=\lim \sigma^n(\mathtt{a})$ exists. Consequently, the principal convergent $\underline{\pi}(w, |\sigma^n(\mathtt{a})|_{\mathrm{symb}})$ converges to $\underline{\delta}(w)$ slowly. Fig. 26 is the graph of $\nu(n)=\nu^{(1)}(n)=\mu(|\sigma^n(\mathtt{a})|_{\mathrm{symb}})$, $2\le n\le 200$ ($\nu(n)<0$ for $2\le n\le 25$ and $n=27$, 29, $|\sigma^{29}(\mathtt{a})|_{\mathrm{symb}}=3^{29}=6.863...\times 10^{13}$; $\nu(n)>0$ for $30\le n\le 200$, $\nu(200)=0.21303...$, $|\sigma^{200}(\mathtt{a})|_{\mathrm{symb}}=3^{200}=2.656...\times 10^{95}$).

Fig. 26 :

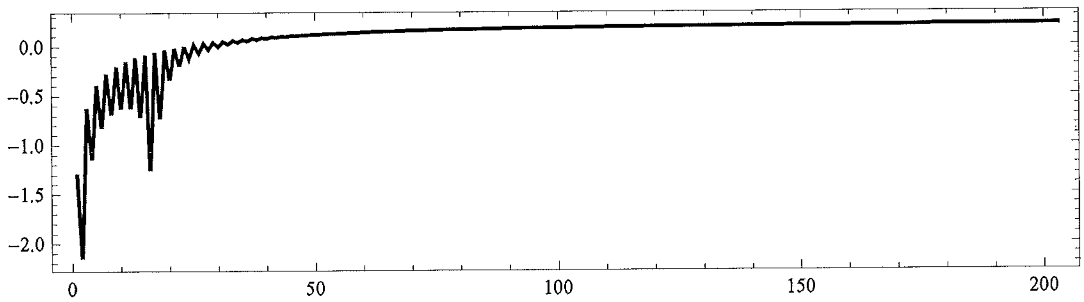


Fig. 27 is (i) proj $_{(0,1)}$(LR(σ^9(a)), (ii) proj $_{(1,0)}$(LR(σ^9(a)), (iii) proj $_{(u,v)}$(LR(σ^9(a)) with (u,v)=(0.56328..., -0.82626...) (u, v are algebraic numbers of degree 16; ||(u,v)||=1, and u, v are roots of a totally real polynomial $80128x^{16}-320512x^{14}+520832x^{12}-440704x^{10}+207152x^8-53728x^6+7288x^4-456x^2+9$). (iv) proj $_{(u,v)}$(LR(σ^9(a)) with (u,v)=$((1/2-\sqrt{-3}/2)/\sqrt{2},-1/\sqrt{2})$=(0.35355...-0.61237... $\cdot\sqrt{-1}$, -0.70710...). The shapes (i), (ii) in Fig 27 are quite different from each other, so that in general, intermediate convergents $\underline{\pi}$(w, n) can not approximate to $\underline{\delta}$(w) well, cf. Fig. 11 (i), (ii) in Example 3. In fact the value μ(n) ($5\leq n\leq N=|\sigma^{12}(a)|_{symb}=531441$) is usually negative and sporadically turns to be positive; it attains not only μ(n)= -783.22..., but also μ(n)=+316.06...; Fig . 28 is the graph of μ(n) for $5\leq n\leq 3^{12}$ (the values $\mu(n)\notin[-1.5, 0.5]$ are excluded).

Fig. 27. (i)

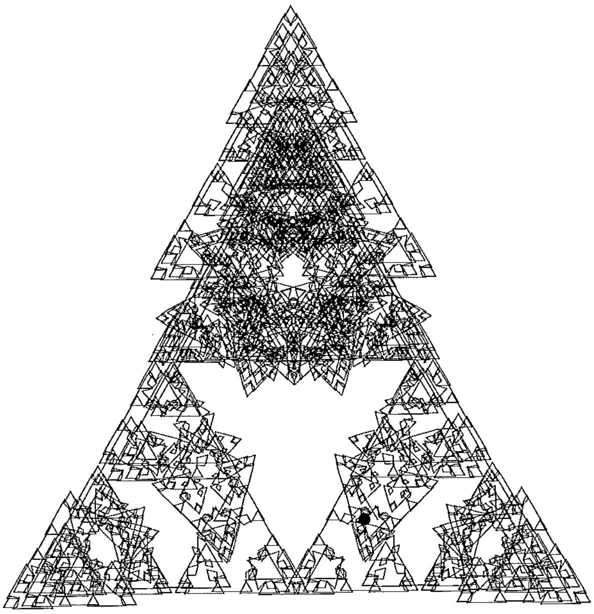


Fig. 27. (ii)

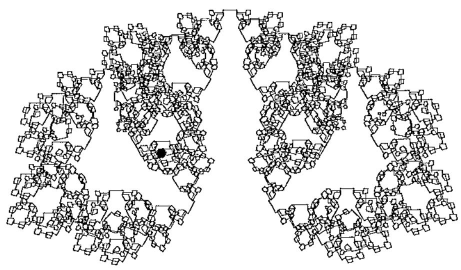


Fig. 27. (iii)

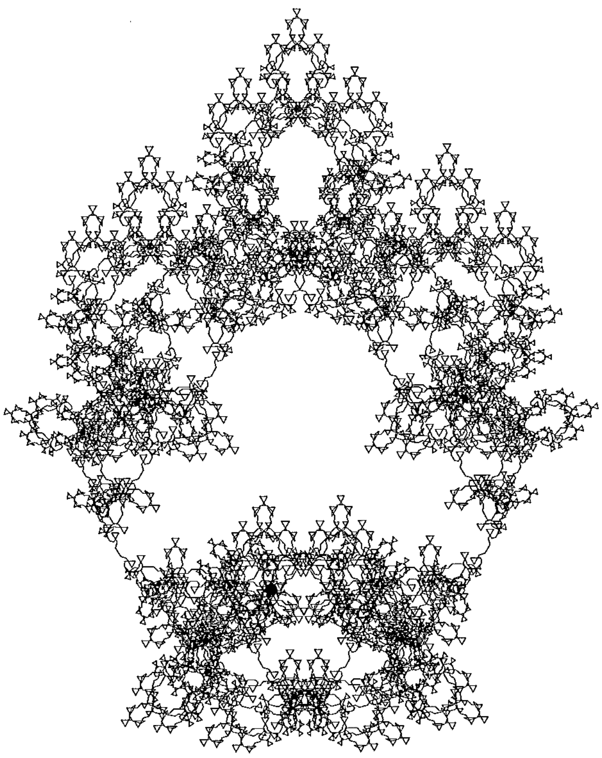


Fig. 27. (iv)

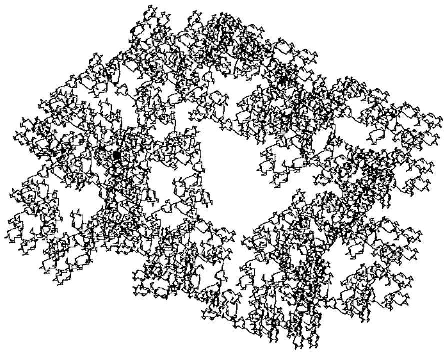


Fig. 28 :

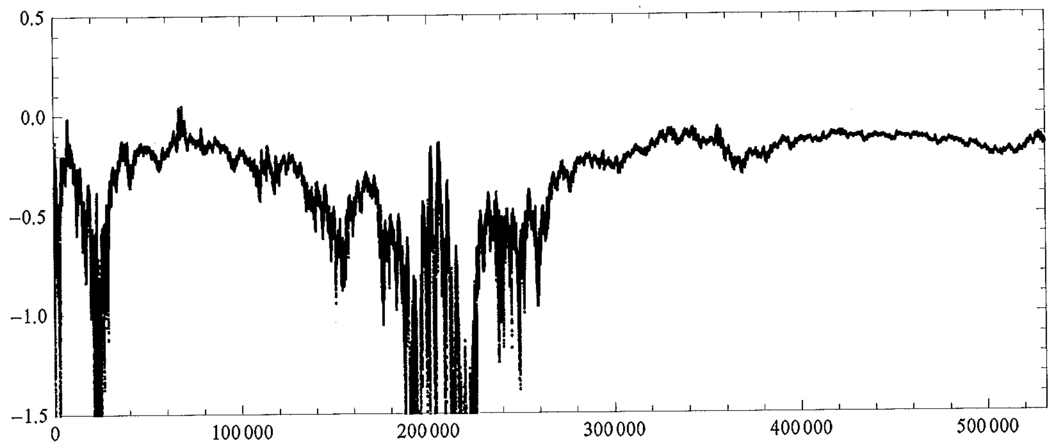


Fig. 29 is proj $_{\underline{\underline{\delta}}(w)}$ (PR(σ^n(a)) for (i) n=9, (ii) n=10, (iii) n=11, (iv) n=12. In view of Fig. 29, we can guess that

$$\lim_{n\to\infty} \mathrm{Diam}(\mathrm{proj}_{\underline{\underline{\delta}}(w)}(\mathrm{PR}(\sigma^n(a))))=\infty,$$

where Diam(S) is the diameter of a set S($\subset\mathbb{C}^s$, or $\mathbb{C}^{s+1}$) in general. We define

$$\mathrm{NR}_{\underline{\underline{\gamma}}}(\sigma^n(a);\kappa):=\kappa^n\cdot \mathrm{proj}_{\underline{\underline{\gamma}}}(\mathrm{PR}(\sigma^n(a)))\quad(\kappa\in\mathbb{C}^\times,\ \underline{\underline{\gamma}}\in\mathbb{C}^{s+1}).$$

Let $\underline{\underline{\xi}}^{\#}$ (resp., $\underline{\underline{\xi}}_{\#}$) be the eigenvector with respect to the dominant eigenvalue $\lambda^{\#}$ (resp., subdominant eigenvalue $\lambda_{\#}$) of M_σ.

(case 1: $\underline{\underline{\gamma}}=\underline{\underline{\xi}}^{\#}$**)** Fig. 30 is the set $\mathrm{NR}_{\underline{\underline{\gamma}}}(\sigma^n(a);\kappa)=\kappa^n\cdot\mathrm{proj}_{\underline{\underline{\gamma}}}(\mathrm{PR}(\sigma^n(a)))$ with (i) n=8, (ii) n=9, (iii) n=10, (iv) n=11, (va) n=12, and (vb) is the same as (va) but different scale, where

$$\underline{\underline{\gamma}}=\underline{\underline{\xi}}^{\#}(=\underline{\underline{\delta}}(w))\text{ and }\kappa=1/\lambda_{\#}.$$

We are not sure that $\kappa=1/\lambda_{\#}$ is the exact value for which the sequence of sets

$$\bigcap_{0\le n<N}\bigcup_{n\le m<\infty}\mathrm{NR}_{\underline{\underline{\gamma}}}(\sigma^m(a);\kappa),\ N=1,2,3,\ldots$$

"converges" to a definite set different$\neq\{\underline{0}\}$, but we have a possibility to define a normalized Rauzy set by

$$(\bigcap_{0\le n<\infty}\bigcup_{n\le m<\infty}\mathrm{NR}_{\underline{\underline{\gamma}}}(\sigma^m(a);\kappa))^{\mathrm{cl}}\tag{11}$$

in some cases by taking a suitable number $\kappa=\kappa(\sigma, a, \underline{\underline{\gamma}})$. For instance,

$$|\kappa|:=\mathrm{Inf}\,\{r\in\mathbb{R}_{>0};\ \lim_{n\to\infty}\mathrm{Diam}(\mathrm{NR}_{\underline{\underline{\gamma}}}(\sigma^n(a);r))=\infty\},$$

or

$$|\kappa|:=\mathrm{Sup}\,\{r\in\mathbb{R}_{>0};\ \lim_{n\to\infty}\mathrm{Diam}(\mathrm{NR}_{\underline{\underline{\gamma}}}(\sigma^n(a);r))=0\}$$

may be a candidate of $|\kappa|$, but we have no idea to give such a formula for arg κ.

(case 2: $\underline{\underline{\gamma}}=\underline{\underline{\xi}}_{\#}$**)** Fig. 31 is proj $_{\underline{\underline{\gamma}}}$(PR($\sigma^n$(a))) with $\underline{\underline{\gamma}}=\underline{\underline{\xi}}_{\#}$ for (i) n=9, (ii) n=10, (iii) n=11, (iv) n=12. Fig. 32 is $\mathrm{NR}_{\underline{\underline{\gamma}}}(\sigma^n(a);\kappa)=\kappa^n\cdot\mathrm{proj}_{\underline{\underline{\gamma}}}(\mathrm{PR}(\sigma^n(a)))$ with $\underline{\underline{\gamma}}=\underline{\underline{\xi}}_{\#}$ and $\kappa=1/\lambda^{\#}$ for (i) n=9, (ii) n=10, (iii) n=11, (iva) n=12. (ivb) is the same as (iva), but enlarged. By such experiments, we may guess that the normalization (11) can be applied in some cases.

Fig. 29.

(i) (ii) (iii) (iv)

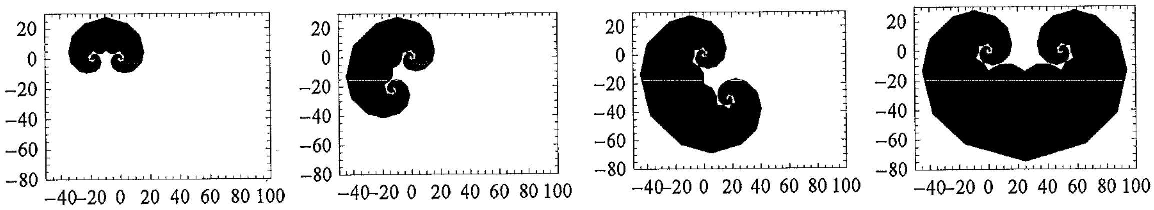
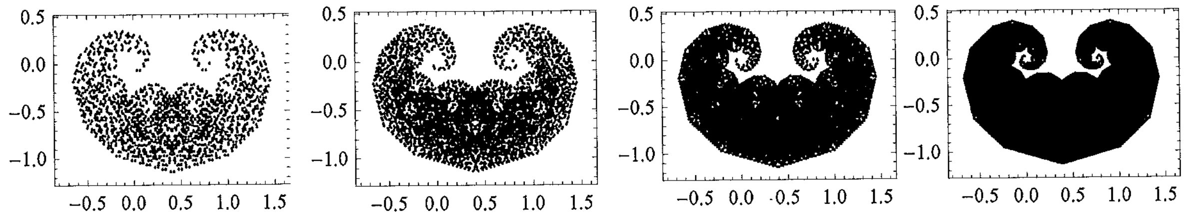


Fig. 30.

(i) (ii) (iii) (iv)



(va) (vb)

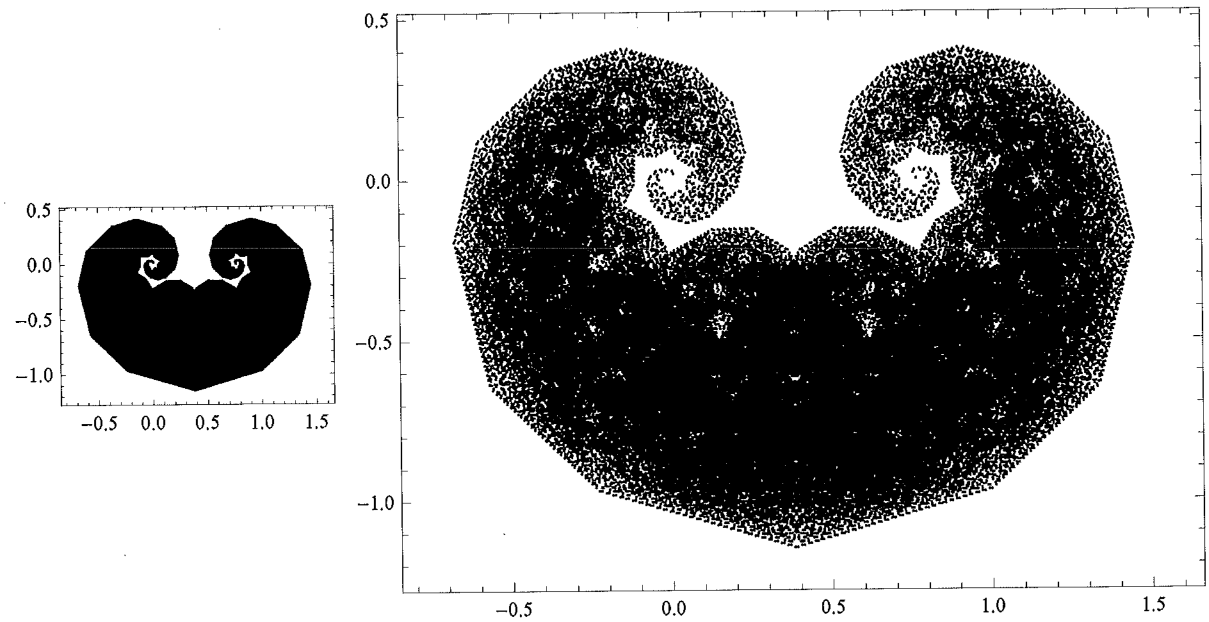


Fig. 31.

(i) (ii) (iii) (iv)

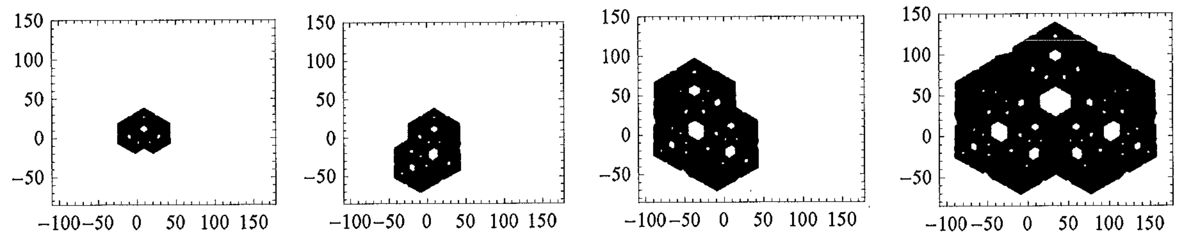
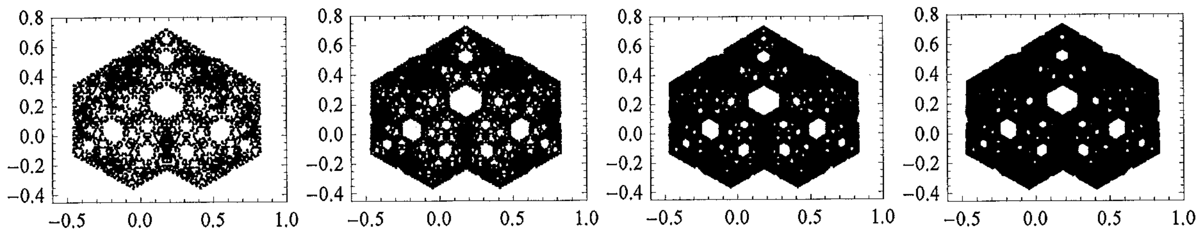
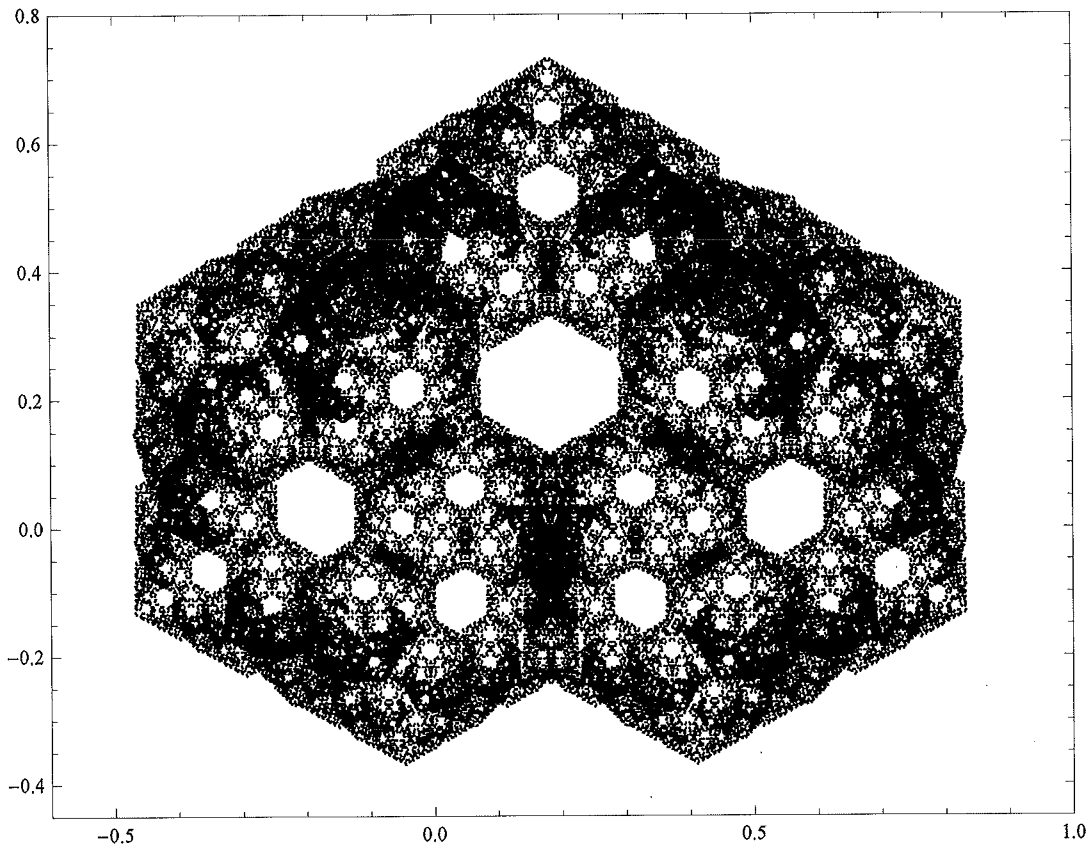


Fig. 32.

(i) (ii) (iii) (iva)



(ivb)



Related to substitutions which are dominant but not Diophantine, we give a conjecture:

Conjecture 16. Let $\sigma\in\mathrm{Sub}_a(\mathscr{A}^\wedge\mathbb{C}^\times)$ be a dominant substitution which is not Diophantine. Suppose that M_σ has an eigenvalue λ with $|\lambda|>1$. Then $\lim_{n\to\infty} \mathrm{Diam}(\mathrm{proj}_{\underline{\delta}(w)}(\mathrm{PR}(\sigma^n(a))))=\infty$ holds.

We aloso give conjectures concerning substitutions σ having a common incidence matrix M_σ:

Conjecture 17. Let $\sigma\in\mathrm{Sub}_a(\mathscr{A}^\wedge\mathbb{C}^\times)$ be a substitution possibly not dominant. Suppose that there exists the direction $\underline{\delta}(w)$ for the fixed point $w=\lim\sigma^n(a)$. Then there exists a substitution $\tau\in\mathrm{Sub}_a(\mathscr{A}^\wedge\mathbb{C}^\times)$ such that

$$M_\tau=M_\sigma \text{ and } \lim_{n\to\infty} \mathrm{Diam}(\mathrm{proj}_{\underline{\delta}(w)}(\mathrm{PR}(\tau^n(a))))=\infty$$

holds.

Conjecture 18. Let $\sigma\in\mathrm{Sub}_a(\mathscr{A}^\wedge\mathbb{C}^\times)$ be a Diophantine substitution. Then there exists a substitution $\tau\in\mathrm{Sub}_a(\mathscr{A}^\wedge\mathbb{C}^\times)$ such that

$$M_\tau=M_\sigma \text{ and } \lim_{n\to\infty} \mathrm{Diam}(\mathrm{proj}_{\underline{\delta}(w)}(\mathrm{PR}(\tau^n(a))))<\infty$$

holds for $w=\lim\sigma^n(a)$.

Notice that $\underline{\delta}(\lim\sigma^n(a))=\underline{\delta}(\lim\tau^n(a))$ holds in Conjectures 17, 18, which follows from $M_\sigma=M_\tau$. The last conjecture could be the most interesting in connection with the construction of (simultaneous) diophantine approximation of the projective direction $\underline{\delta}(w)$ by intermediate convergents $\underline{\pi}(w, n)$.

§8. Varieties

For the definition of Rauzy set of a substitution, the multiplicative property $M_{\sigma\tau}=M_\sigma M_\tau$, cf. Lemma 1 in Section 3 may be important, since almost everything comes from the multiplicity as we have already seen. In this sense, the property (2) is the most important. We may modify the local compulsion (1) such that the property (2) still holds.
There are so many possibilities in definitions of substitutions and local compulsions.

We can consider a substitution given by

$$\sigma(a_j):=b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}}\ (b_{i,j}\in\mathscr{A},\ z_{i,j}\in\mathbb{C}^\times\ (1\le i\le k_j,\ 0\le j\le s))$$

together with some other local compulsion. Recall the definition (4):

$$\sigma(a_j^{z}):=[\sigma(a_j)]^{z}(=[b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}}]^{z})\ (j=0,1,\ldots,s)$$

in Section 3, and the compulsion given by (1):

$$[w]^{z}:=w_1^{z\cdot z_1}w_2^{z\cdot z_2}\cdots w_n^{z\cdot z_n}\ (\forall z\in\mathbb{C}^\times).$$

We can give alternative definitions of substitutions by making restriction to the local compulsion (1) only for "small" $z\in\mathbb{C}$. For instance

$$\begin{aligned}\sigma(a_j^{8/3})&=\sigma(a_j^{1+1+2/3})\\ &:=(b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}})(b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}})[b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}}]^{2/3}\\ &=b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}}b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}}b_{1,j}^{2/3\cdot z_{1,j}}b_{2,j}^{2/3\cdot z_{2,j}}\cdots b_{k_j,j}^{2/3\cdot z_{k_j,j}}.\end{aligned}$$

In general, for $z=re^{\sqrt{-1}\,\theta}$ ($r>0$, $-\pi<\theta\le\pi$), we can give a "polar definition" of substitutions by

$$\begin{aligned}\sigma(a_j^{z})&=\sigma(a_j^{\lfloor r\rfloor e^{\sqrt{-1}\,\theta}+\langle r\rangle e^{\sqrt{-1}\,\theta}})\\ &:=[(b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}})^{(\lfloor r\rfloor)}]^{e^{\sqrt{-1}\,\theta}}[b_{1,j}^{z_{1,j}}b_{2,j}^{z_{2,j}}\cdots b_{k_j,j}^{z_{k_j,j}}]^{\langle r\rangle e^{\sqrt{-1}\,\theta}},\end{aligned}$$

where $\lfloor r \rfloor$ (resp., $<r>$) is the integer part of r (resp., fractional part of r), and we mean by $(w)^{(n)}$ the word obtained by concatenating n copies of an identical word w. It is clear that such a new definition still satisfies the property (2). Instead of the restriction to the local compulsion, we may restrict substitutions by considering only substitutions defined by

$$\sigma(a_j):=b_{1,j}{}^{z_{1,j}}b_{2,j}{}^{z_{2,j}}\cdots b_{k_j,j}{}^{z_{k_j,j}}\in(\mathscr{A}^{\wedge}D^{\times})^{+}\ (b_{i,j}\in\mathscr{A},\ z_{i,j}\in D^{\times}\ (1\leq i\leq k_j,\ 0\leq j\leq s))$$

where

$$D^{\times}:=\{z\in\mathbb{C};\ 0<|z|\leq 1\},$$

is the unit disc, and $(\mathscr{A}^{\wedge}D^{\times})^{+}$ is the set of nonempty finite words over $\mathscr{A}^{\wedge}D^{\times}:=\{a^{z};\ a\in\mathscr{A},\ z\in D^{\times}\}$. We have been mainly concerned with substitutions of this type related to experiments given in Section 7. The fixed point of such a substitution $\sigma\in\mathrm{Sub}_a(\mathscr{A}^{\wedge}\mathbb{C}^{\times})$ becomes a symbolically infinite word of the form

$$a^{z_1}w_2{}^{z_2}\cdots w_n{}^{z_n}\cdots \text{ with } z_n\in\mathrm{Monoid}(\sigma)\subset D^{\times},\ z_1=1,$$

where Monoid(σ) is the monoid (with a unit 1 with respect to the usual multiplication of complex numbers) generated by the numbers$\in Z(\sigma)$ (the set Monoid(σ) is at most countably infinite, and Monoid(σ)=$\{1\}$ holds for some usual substitutions σ over $\mathscr{A}$). In particular, substitutions over $\mathscr{A}^{\wedge}\{z\in\mathbb{C};\ |z|=1\}$ are of special interest. In the case where Monoid(σ) $=\{\pm 1\}$, we may modify the local compulsion by

$$\sigma(a_j{}^{-1})=[\sigma(a_j)]^{-1}=[b_{1,j}{}^{z_{1,j}}b_{2,j}{}^{z_{2,j}}\cdots b_{k_j,j}{}^{z_{k_j,j}}]^{-1}$$
$$:=b_{k_j,j}{}^{-z_{k_j,j}}b_{k_{j-1},j-1}{}^{-z_{k_{j-1},j-1}}\cdots b_{1,j}{}^{-z_{1,j}}.$$

Then we can consider an endomorphism σ on the free group generated by $\mathscr{A}$, and an extension of σ to the infinite words over $\mathscr{A}^{\wedge}\mathbb{Z}$ under the identification of a finite/infinite word u and all the words v obtained by applying the law of exponents

$$a^{x}a^{y}=a^{x+y}\ (\forall a\in\mathscr{A},\ x,y\in\mathbb{Z})$$

everywhere appearing as a subword $a^{x}a^{y}$ in the expression of the word u.

We can give a "cartesian definition" of $[u]^{z}$ for $z=x+y\sqrt{-1}$, for instance

(11) $$[u]^{z}=u^{\lfloor x\rfloor+\lfloor y\rfloor\sqrt{-1}+<x>+<y>\sqrt{-1}}:=u^{(\lfloor x\rfloor)}[u^{(\lfloor y\rfloor)}]^{\sqrt{-1}}[u]^{<x>}[u]^{<y>\sqrt{-1}}$$

Substitutions according to the last definition also satisfy the property (2) as well as previous ones. If $xy\neq 0$, we may modify (11) according to the "finite Sturmian word" coming from the simple continued fraction expansion of y/x. For instance, if $y/x=(1+\sqrt{5})/2$ and $8<x<9$, we define

$$[u]^{z}:=uu^{\sqrt{-1}}uuu^{\sqrt{-1}}uu^{\sqrt{-1}}uuu^{\sqrt{-1}}uuu^{\sqrt{-1}}[u]^{<x>}[u]^{(y-5)\sqrt{-1}}.$$

Notice that the word on the right-hand side can be written as

$$\mathrm{ABAABABAABAAB}\ [u]^{<x>}[u]^{(y-5)\sqrt{-1}}\ (A:=u,\ B:=u^{\sqrt{-1}}),$$

and ABAABABAABAAB is a prefix of the Fibonacci word. By such a modification, we can expect that the quality of the diophantine approximation of the projective direction $\underline{\delta}(w)$ by intermediate convergents $\underline{\pi}(w,n)$ becomes better for a fixed point w of σ in general. Using the ring of algebraic integers of an imaginary quadratic number field instead of the ring/lattice $\mathbb{Z}[\sqrt{-1}]$, we can give some modification of (11). Such modifications will be interesting in conection with "relative continued fraction" over a quadratic number field including the so called complex continued fractions, cf. [H], [Ro], [KST], [TY1].

Thus, there are many possibilities for definitions of substitutions satisfying the multiplicative property. The incidence matrix M_σ, and the principal convergents remain invariant under such modifications of a substitution σ, while in general, intermediate convergents depend on the modifications as well as Rauzy sets. It will be interesting to see the difference of the Rauzy sets and intermediate convergents coming from a substitution under many varieties of the local compulsions satisfying (2). Related to substitutions having the same incidence matrix.

We may give, in some contexts, some other definitions of $[w]^{z}$ different from (1) which do not satisfy (2), and extend formal language theory to a complex powered world.

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